TWO-SCALE COMPOSITE FINITE ELEMENT METHOD FOR PARABOLIC PROBLEMS IN CONVEX AND NONCONVEX POLYGONAL DOMAINS

by

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February, 2019

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A thesis submitted

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

by

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DECLARATION

It is certified that the work contained in this thesis entitled **"Two-scale composite finite element method for parabolic problems in convex and nonconvex polygonal domains"** has done by me, under the supervision of **Dr. Rajen Kumar Sinha**, Professor, Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

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CERTIFICATE

It is certified that the work contained in this thesis entitled **"Two-scale composite finite element method for parabolic problems in convex and nonconvex polygonal domains"** by **Tamal Pramanick (136123008)**, a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

February, 2019

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Dedicated

to

my Parents, Dida and beloved Mamu

Acknowledgements

There is no way I can thank everybody who helped me in this doctoral work, but this thesis would not have been possible without the help and support of the kind peoples around me.

In the first place, I want to express my deepest gratitude towards my thesis supervisor Prof. Rajen Kumar Sinha for his valuable guidance, scholarly inputs and consistent encouragement during my research period in Indian Institute of Technology (IIT) Guwahati. I would like to thank him for inspire me, sometimes forcing, to look at my research work in different ways and opening my mind. His support was essential to my research journey. He discussed all ideas in this thesis, read all the words and provided the foundation to explore the subject with great freedom. For being such a generous and well behaved person, he always listened my countless queries with a great patience and provided insightful comments which helped to make my Ph.D experience productive and stimulating. I admire the tremendous amount of effort he has put in this thesis from his busy schedule of different administrative responsibilities. His unflinching courage and conviction always inspire me, and I hope to continue to work with his noble thoughts.

My gratitude towards Prof. Frédéric Hecht, Université Pierre et Marie Curie, Paris, France is beyond words. In my research period, while I was staying in IIT Bombay for attending CIMPA workshop, he helped me tremendously to understand the numerical experiments in a better way. Whenever I contacted him through e-mail, he was always readily available to respond. Thank you Sir for your immense support in my research.

I am indebted and thankful to the doctoral committee members Prof. R. Alam, Prof. S. Natesan and Prof. B. Deka for their insightful comments and suggestions on my work. I express my sincere thanks to all the faculty members of the Department of Mathematics for their help and cooperations. I would also like to thank IIT Guwahati for facilities provided to me during my research work. I am most grateful to the Ministry of Human Resource Development, Government of India, for providing me financial assistance for the completion of my thesis work.

My sincere thanks to Prof. S. N. Bora (former Head of the Department) and Prof. N. Selvaraju (present Head of the Department) for providing such a friendly atmosphere and all the facilities for my research work. I owe my sincere thanks to Prof. A. K. Chakrabarty for his dedicated teaching procedure which helped me to learn many things. I thank him for his tremendous patience to listen all my queries. I am lucky to have some courses taught by Prof. A. Saikia, Prof. D. C. Dalal, Prof. B. K. Sarma and Prof. S. Bora during my course work in Ph.D program.

I thank our technical superintendent Mr. Shantanu Majumdar and Mr. Pranpratim Borgohain and office staff members Mr. Sridhar Samal, Mr. Phatik Kumar and Mr. Saurav Choudhury of the department, for their assistance in various ways during my research period.

I take this opportunity to convey my gratitude to my friends and fellow research scholars for all their timely help, encouragement and support during this period. Some individuals are worth mentioning. I articulate my thankfulness to my seniors and friends Abhishek, Arnab, Anirban, Debopam, Swarup, Nasim, Hiranmoy, Sougata, Swapnendu, Ankur, Anirudha, Swarup, Maneesh, Ashish, Ranjan, Ramesh, Koyel and Balasubramani for all their encouragement and support during this period. My special appreciation goes to my close friends Debkumar, Sonjoy, Madhusudan, Dipankar, Sangram, Shantiram and Nilay with whom I have shared some of the best moments of my life.

I would like to convey my gratitude to my seniors Dr. Murali M. Reddy and Dr. Jhuma Sengupta and to my friends Tanushree Ray and Pratibha Shakya. My juniors Mr. Shantiram Mahata and Mr. Ram Manohar also deserve special gratitude.

I am highly indebted to my family members especially my Baba (Mr. Tushan Pramanick) and Maa (Mrs. Archana Pramanick) for their continuous love, encouragement and inspiration throughout my life. Words can never be enough to express my deepest sense of gratitude towards them. I owe everything to them and this thesis is dedicated to them. The endless thanks goes to my Mamu and Dida for all the blessings they have showered onto me. Besides all my well wishers who have helped me knowingly and un-

knowingly in the successful completion of this work, my heartfelt gratitude goes to them. Above all, I would like to thank God Almighty for giving me the strength, knowledge, ability and opportunity to undertake this research study and to persevere and complete it satisfactorily.

February, 2019

With regards,

(Tamal Pramanick)

Department of Mathematics

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"Strength if life,
weakness is death."
—*Swami Vivekananda*

"Once you start working on something, don't
be afraid of failure and don't abandon it.
People who work sincerely are the happiest."
—*Chanakya (Kautilya)*

"Arise! Awake! and stop not
until the goal is achieved."
—*Swami Vivekananda*

Abstract

The main objective of this thesis is to study *a priori* error analysis of the two-scale composite finite element (CFE) method for parabolic initial-boundary value problems (IBVPs) in two-dimensional convex and nonconvex polygonal domains. When the physical domain is nonconvex or very complicated, the standard finite element method (FEM) requires to generate finite element mesh that resolves the domain boundary. As a result, the degrees of freedom of such finite element space are distributed in a nonoptimal way with respect to the approximation quality which drastically increases the minimal dimension of the finite element space. Whereas, the CFE discretizations are based on the two-scale grid refinement: In the interior of the domain at a proper distance from the boundary, the solution is approximated by a coarse-scale parameter H whereas the near-boundary triangles are discretized by a fine-scale parameter h which approximates the Dirichlet boundary conditions. The coarse-scale grid $\mathcal{T}_H$ contains the degrees of freedom whereas the fine-scale grid $\mathcal{T}_h$ adaptively resolves the boundary Γ that contains the slave nodes only. The basic idea of the CFE procedure is to work with fewer degrees of freedom by allowing finite element mesh to resolve the domain boundaries and to preserve the asymptotic order convergence on coarse-scale mesh. In contrast to the standard finite elements, CFE method uses the minimal dimension of the approximation space. These new class of finite elements can be thought of as a generalization of standard finite elements by allowing the approximation of the domain boundaries in a flexible adaptive manner. This is especially very advantageous for problems on complicated domains.

The present dissertation is commenced with general introduction to the respective field along with the detailed overview of CFE discretizations. Additionally, the background knowledge and motivations of *a priori* error analysis for the two-scale CFE approximation of parabolic IBVPs in two-dimensional convex and nonconvex polygonal domains are discussed. To begin with, we first study *a priori* estimates in both $L^\infty(L^2)$ and $L^\infty(H^1)$-norms for the spatially semidiscrete and fully discrete CFE approximation of the linear parabolic problem in a convex domain with smooth initial data. The estimates of the associated elliptic or Ritz projection in the framework of CFE method play an important role in the error analysis. We derive optimal order convergence (up to logarithmic terms) for smooth initial data. Then, we concentrate on the nonsmooth data error analysis for the homogeneous linear parabolic equa-

tion in a convex domain. Optimal order error estimates (up to logarithmic terms) with respect to space discretization are shown to hold for positive time even for nonsmooth initial data. The eigenfunctions expansion related to the elliptic operator and the rational approximation for exponential are the key technical tools used in nonsmooth data error analysis.

We next proceed to the *a priori* error analysis for the spatially semidiscrete and fully discrete CFE approximation of linear parabolic problem in a nonconvex domain for both smooth and nonsmooth initial data. Compared to the error estimates for convex domain, the rates of convergence in nonconvex domain is reduced for both smooth and nonsmooth data. Furthermore, we consider the CFE error analysis for nonlinear parabolic problems in nonconvex domains. Both the spatially semidiscrete and fully discrete schemes are analyzed and the related error estimates are derived. Finally, we focus on *a priori* error estimates in the $L^2(L^2)$-norm for the spatially semidiscrete and fully discrete CFE approximation of parabolic problem with measure data in time for both convex and nonconvex domains. The solution of this kind of problem exhibits low regularity which introduces some difficulties in both theory and numerics of CFE method. An effort has been made to investigate the CFE error analysis and derive convergence properties for both convex and nonconvex domains.

We perform extensive numerical experiments of the proposed CFE method for both convex and nonconvex domains to support our theoretical results. It is shown that the number of degrees of freedom for CFE method is much less compared to the standard FEM, which substantially reduces the computational effort.

Contents

NOMENCLATURE

BVP	Boundary value problem
IBVP	Initial-boundary value problem
PDE	Partial differential equation
FEM	Finite element method
CFE	Composite finite element
ROC	Rate of convergence
$\mathbb{R}^n$	n-th dimensional Euclidean space
Ω	Bounded convex or nonconvex polygonal domain
Γ	Boundary of Ω
u	Exact solution of PDE
u_0	Initial function
f	Non-homogeneous term
$L^p(\Omega)$, $1 \leq p \leq \infty$	Standard Lebesgue space of order p over the measurable set Ω
$\\|\cdot\\|_{L^p(\Omega)}$	Norm on $L^p(\Omega)$
$(\cdot,\cdot)_\Omega$	Standard $L^2(\Omega)$ inner product
$W^{m,p}(\Omega)$	Standard Sobolev space of order (m,p) over the measurable set Ω
$\\|\cdot\\|_{m,p}$	Norm on $W^{m,p}(\Omega)$
$H^m(\Omega)$	Hilbert space $W^{m,2}(\Omega)$
$H_0^1(\Omega)$	Space of functions in $H^1(\Omega)$ that vanish on the boundary of Ω
$L^p(0,T;\mathbf{B})$	The standard Bochner space with $\mathbf{B}$ is a Banach space $(1 \leq p \leq \infty)$
$\\|\cdot\\|_{L^p(0,T;\mathbf{B})}$	Norm on $L^p(0,T;\mathbf{B})$
$D^{\widetilde{\gamma}}\phi$	One of the partial derivative of order $\|\widetilde{\gamma}\|$ of $\phi : \mathbb{R}^n \to \mathbb{R}$
$C^m(\overline{\Omega})$	Space of functions with continuous derivatives up to and including order m in $\overline{\Omega}$

$C_0^m(\Omega)$	Space of all $C^m(\Omega)$ functions with compact support in Ω
$C_0^\infty(\Omega)$	Space of all infinitely differentiable functions with compact support in Ω
$\mathrm{supp}(\phi)$	Support of ϕ
$\mathcal{A}(\cdot,\cdot)$	Bilinear form corresponding to the elliptic operator $-\Delta$
α_0	Continuity constant for $\mathcal{A}(\cdot,\cdot)$
β_0	Coercivity constant for $\mathcal{A}(\cdot,\cdot)$
$\mathcal{T}_{H,h}$	Two-scale grid discretization for the domain Ω
$\vartheta_{H,h}$	The set of grid points corresponding to $\mathcal{T}_{H,h}$
$\mathcal{T}_H$	Initial overlapping coarse grid triangulations
ϑ_H	The set of grid points corresponding to $\mathcal{T}_H$
$\mathcal{T}_h$	Finer grid triangulations
$\mathcal{T}_H^{\mathrm{in}}$	Inner grid of $\mathcal{T}_{H,h}$
ϑ_{dof}	The set of vertices (degrees of freedom) corresponding to $\mathcal{T}_H^{\mathrm{in}}$
$\mathcal{T}_\Gamma$	A subset of $\mathcal{T}_H$ containing all near-boundary triangles
$\vartheta_{\mathrm{slave}}$	Set of all slave nodes $\vartheta_{\mathrm{slave}} := \vartheta_{H,h} \setminus \vartheta_{\mathrm{dof}}$
$\mathcal{E}$	Extrapolation operator
$\mathscr{E}$	Extension operator
$\mathbf{\Phi}$	Grid function
K	Triangle (closed)
τ	To obtain τ, each K is splitted into four triangles by joining the mid-points of the edges of K
$\mathrm{diam}(K)$	Longest side of K
$\mathrm{int}(K)$	Interior of K
$\mathrm{sons}(K)$	Set of τ
n_K	$\sharp\,\mathrm{sons}(K)$
Ω^{in}	$\mathrm{int}\left(\bigcup_{K\in\mathcal{T}_H^{\mathrm{in}}} K\right)$

h_K	$\mathrm{diam}(K)$
H	$\max(h_K)$ for $K \in \mathcal{T}_H$
h_τ	$\mathrm{diam}(\tau)$
h	$\min(h_\tau)$ for $\tau \in \mathcal{T}_{H,h},\ h \le H$
J	The time interval $(0, T]$ with $T < \infty$
J_n	n-th subinterval of $[0, T]$
k_n, k	Variable time step, constant time step
Ω_T, Γ_T	$\Omega \times J$, $\Gamma \times J$
Δ_x	For $x \in \vartheta_{\text{slave}}$, $\Delta_x \in \mathcal{T}_H^{\text{in}}$ has minimum distance to x
x^Γ	For $x \in \vartheta_{\text{slave}}$, $x^\Gamma \in \Gamma$ has minimum distance to x
$R_{\tau,x,y}$	Minimal rectangle containing x^Γ, y^Γ and τ with one side being parallel to $\overline{x^\Gamma y^\Gamma}$
$\mathbf{G}(K)$	Set of vertices of a triangle K
$\mathbb{P}_\ell$	Space of polynomials of degree $\le \ell$
Φ_K	Linear function which interpolates $\mathbf{\Phi}$ in the vertices of K
S	Continuous, piecewise linear finite element space with respect to the mesh $\mathcal{T}_{H,h}$
S^{in}	Continuous, piecewise linear finite element space with respect to the mesh $\mathcal{T}_H^{\text{in}}$
S^{CFE}	Composite finite element space with respect to the mesh $\mathcal{T}_{H,h}$
R^{CFE}	Elliptic or Ritz projection onto S^{CFE}
P^{CFE}	L^2-projection onto S^{CFE}
σ_{dist}	The parameter control the width of the neighborhood of Γ
σ_e	The parameter control the distance from an inner triangle Δ_x to a slave node x
σ_{uni}	The constant describes the local quasi-uniformity of the initial overlapping mesh $\mathcal{T}_H$ near the boundary

σ_{ext}	The constant is related to the extension operator $\mathscr{E}$
C_P	Poincaré's constant
$C_{\text{ol},1}$, $C_{\text{ol},2}$	Overlapping constants
CFE_{err1}, CFE_{err2}	CFE errors in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms
FE_{err1}, FE_{err2}	Standard finite element errors in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms
CFE_{err3}	CFE errors in the $L^2(L^2)$-norm
FE_{err3}	Standard finite element errors in the $L^2(L^2)$-norm

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1

Introduction

The numerical approximation of parabolic partial differential equations (PDEs) posed on a convex and nonconvex polygonal domain, are of vital importance in science and engineering applications. The main objective of the present thesis is to study *a priori* error analysis for the two-scale composite finite element (CFE) approximations of parabolic problems in both convex and nonconvex polygonal domains. This chapter introduces the problems along with the necessary notations and preliminaries. It also includes a brief survey of the relevant literature and the motivational background for the present study. Finally, the organization of the thesis is presented in the last section of this chapter.

1.1 Description of the Problems

In this section, we briefly discuss the model problems which are considered in this thesis.

Linear parabolic problems in convex and nonconvex domains. Consider the linear parabolic problems of the form

$$
\begin{aligned}
u_t - \Delta u &= f \quad \text{in } \Omega,\ t > 0,\\
u &= 0 \quad \text{on } \Gamma,\ t > 0, \qquad (1.1)\\
u(\cdot, 0) &= u_0 \quad \text{in } \Omega,
\end{aligned}
$$

where Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary Γ having a finite length. For simplicity, we assume that Ω is a polygonal domain, but the boundary of the domain may still have a very complicated shape. Here $u = u(x,t)$, $f = f(x,t)$, u_t denotes $\frac{\partial u}{\partial t}$, and the Laplacian is denoted by $\Delta = \sum_{j=1}^{2} \partial^2/\partial x_j^2$. The initial data u_0 is a function of x only and is assumed to be smooth.

We also consider the initial-boundary value problem (IBVP) for a homogeneous parabolic equation, where the initial data is assumed to be nonsmooth. Thus, the following model problem:

$$
\begin{aligned}
&u_t = \Delta u \quad \text{in } \Omega,\ t > 0, \\
&u = 0 \quad \text{on } \Gamma,\ t > 0, \quad \text{with} \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega,
\end{aligned} \tag{1.2}
$$

is considered and analyzed for both convex as well as nonconvex polygonal domains when the given data $u_0 \in L^2(\Omega)$.

Nonlinear parabolic problems in nonconvex domains. Let Ω be a nonconvex polygonal domain in $\mathbb{R}^2$ with boundary Γ. Set $\Omega_T = \Omega \times J$ and $\Gamma_T = \Gamma \times J$ with $J := (0, T]$, $T < \infty$. As a model problem, we consider the following nonlinear parabolic problem:

$$
\begin{aligned}
u_t - \nabla \cdot (a(u)\nabla u) &= f(u) \quad \text{in } \Omega_T, \\
u &= 0 \quad \text{on } \Gamma_T, \\
\text{with } u(\cdot, 0) &= u_0 \quad \text{in } \Omega,
\end{aligned} \tag{1.3}
$$

where $u = u(x, t)$. Further, we assume that the functions a and f are smooth on $\mathbb{R}$ such that

$$
0 < \widehat{\mu} \le a(u) \le M, \quad |a'(u)| + |f'(u)| \le \widehat{B} \quad \text{for } u \in \mathbb{R},
$$

where $\widehat{\mu}$, M and $\widehat{B}$ are some positive constants.

Parabolic problems with measure data in time in convex and nonconvex domains. Let Ω be a bounded polygonal domain (may be convex or nonconvex) in $\mathbb{R}^2$ with boundary Γ. Consider the following model IBVP:

$$
\begin{aligned}
u_t - \Delta u &= \mu \quad \text{in } \Omega_T, \\
u &= 0 \quad \text{on } \Gamma_T, \\
u(\cdot, 0) &= u_0 \quad \text{in } \Omega,
\end{aligned} \tag{1.4}
$$

where $u = u(x, t)$, $u_t = \frac{\partial u}{\partial t}$, $\Delta = \sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2}$ the Laplacian and the initial data u_0 is a function of x only. The source term $\mu = \sigma\omega$, where σ and ω are given functions such that $\sigma \in C([0, T]; L^2(\Omega))$ and $\omega \in \mathcal{M}[0, T]$. Here $\mathcal{M}[0, T]$ is the space of real and regular Borel measures in $[0, T]$, which can be defined as the dual space of $C[0, T]$ with

$$
\|\mu\|_{\mathcal{M}[0,T]} = \sup \left\{ \int_0^T v \, d\mu \ : \ v \in C[0, T] \ \text{and} \ \|v\|_{C[0,T]} \le 1 \right\}.
$$

Parabolic PDEs in convex and nonconvex domains appear in many physical phenomena such as modelling of heat conduction in composite materials, environment modelling, porous media flow and modelling of complex technical engines. Rigorous discussions on heat conduction in chip design, elastic behaviour of composite materials, compression of poro-viscoelastic media can be found in the articles of Coleman [26], Coleman and Gurtin [27], Gao *et al.* [44], Habetler and Schiffman [49], Kosmopoulos and Keller [59], Pegoretti *et al.* [75] and in the respective references therein. Furthermore, several works involving nonlinear parabolic equations in nonconvex polygonal domains can be found in the articles of Gao *et al.* [43] along with that of Li and Zhang [64]. The regularity analysis for the solution of some linear and nonlinear models in a polygonal domain has been extensively studied by Grisvard [48] and Kweon [61, 62]. Parabolic equations with measure data in time have varied applications. One of the most important applications for parabolic equations with measure data in time is related to the first order optimality conditions of parabolic optimal control problems with pointwise state constraints, see e.g., Casas [19], Lions [65], Meidner *et al.* [72] and the references quoted therein. In general, for parabolic optimal control problems with pointwise state constraints, the Lagrange multiplier μ associated to the state constraints belongs to $\mathcal{M}(\overline{\Omega}_T)$, where $\mathcal{M}(\overline{\Omega}_T)$ is the space of regular Borel measures on $\overline{\Omega}_T$, $\mu_{\Omega_T} := \mu|_{\Omega_T}$, $\mu_{\Gamma_T} := \mu|_{\Gamma_T}$ and $\mu_T := \mu|_{\overline{\Omega}\times\{T\}}$ (cf. Deckelnick and Hinze [30]). However, in the case where only pointwise in time state constraints are imposed, the Lagrange multiplier μ appears to be a measure only in time, and hence it can be decomposed as $\mu = \sigma\omega$, where σ and ω are already mentioned.

It is evident from the discussion that the importance of the aforementioned parabolic equations is noteworthy as they signify various physical phenomena. It is worthy of mentioning that the topic of *a priori* error analysis for the two-scale CFE method with respect to these equations remained unexplored. Therefore, an attempt is made in this thesis to study *a priori* error analysis for the two-scale CFE approximations of parabolic problems.

1.2 Some Notations and Terminologies

In this section, we introduce some standard notations, function spaces and recall some basic definitions and results which will be used throughout the thesis. All functions considered here are real-valued. For the purpose of including notations, we assume that Ω is a polygonal domain in $\mathbb{R}^n$ with boundary Γ. Let $x = (x_1, x_2, \cdots, x_n)$ be an n-tuple with $dx := dx_1 dx_2 \cdots dx_n$. The order of $\widetilde{\gamma}$, where $\widetilde{\gamma} = (\widetilde{\gamma}_1, \widetilde{\gamma}_2, \cdots, \widetilde{\gamma}_n)$ with $\widetilde{\gamma}_i \in \mathbb{Z}_+$ (the set of nonnegative integers), is denoted by $|\widetilde{\gamma}| := \widetilde{\gamma}_1 + \widetilde{\gamma}_2 + \cdots + \widetilde{\gamma}_n$. Then the $\widetilde{\gamma}$th derivative of ϕ denoted by $D^{\tilde{\gamma}}\phi$ is defined by

$$D^{\tilde{\gamma}}\phi := \frac{\partial^{|\tilde{\gamma}|}\phi}{\partial x_1^{\tilde{\gamma}_1}\partial x_2^{\tilde{\gamma}_2}\cdots\partial x_n^{\tilde{\gamma}_n}}.$$

The support of a function ϕ on Ω is denoted by supp(ϕ) is defined by

$$\text{supp}(\phi) := \overline{\{x \in \Omega \mid \phi(x) \neq 0\}}.$$

We say that ϕ has a compact support in Ω if supp(ϕ) is a compact set in Ω.

Now we introduce the following function spaces which we shall refer frequently. For any nonnegative integer m, $C^m(\overline{\Omega})$ is defined as the space of m times continuously differentiable functions on $\overline{\Omega}$. We write $C_0^m(\Omega)$ to denote the space of all $C^m(\Omega)$ functions with compact support in Ω, and $C_0^\infty(\Omega)$ is the space of all infinitely differentiable functions with compact support in Ω. Given a Lebesgue measurable set $\Omega \subset \mathbb{R}^n$, the Lebesgue spaces $L^p(\Omega)$, $1 \leq p < \infty$, refer to the linear space of equivalence classes of measurable functions ϕ on Ω such that $\int_\Omega |\phi(x)|^p dx$ exists and is finite. The space $L^p(\Omega)$, $1 \leq p \leq \infty$ is equipped with the norm

$$\begin{aligned} \|\phi\|_{L^p(\Omega)} &:= \left(\int_\Omega |\phi(x)|^p dx\right)^{1/p}, \quad 1 \leq p < \infty, \\ \text{and} \quad \|\phi\|_{L^\infty(\Omega)} &:= \operatorname*{ess\,sup}_{x\in\Omega}|\phi(x)| < \infty, \quad p = \infty. \end{aligned}$$

When $p = 2$, $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(\phi, \psi)_\Omega := \int_\Omega \phi(x)\psi(x)dx.$$

We now introduce the notion of Sobolev spaces. For an integer $m \geq 0$ and $1 \leq p \leq \infty$, the Sobolev space $W^{m,p}(\Omega)$ is defined as a linear space of functions (or equivalence class of functions) in $L^p(\Omega)$ whose distributional derivatives of order up to m are also in $L^p(\Omega)$, i.e.,

$$W^{m,p}(\Omega) := \{\phi \mid D^{\tilde{\gamma}}\phi \in L^p(\Omega) \;\text{ for }\; 0 \leq |\tilde{\gamma}| \leq m\}.$$

The space $W^{m,p}(\Omega)$ is endowed with the norm

$$\begin{aligned} \|\phi\|_{m,p} &:= \left(\sum_{0\leq|\tilde{\gamma}|\leq m} \|D^{\tilde{\gamma}}\phi\|^p_{L^p(\Omega)}\right)^{1/p}, \quad 1 \leq p < \infty, \\ \|\phi\|_{m,\infty} &:= \max_{0\leq|\tilde{\gamma}|\leq m} \|D^{\tilde{\gamma}}\phi(x)\|_{L^\infty(\Omega)}, \quad p = \infty. \end{aligned}$$

In particular, for $p = 2$, we denote the spaces $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and $H^0(\Omega) = L^2(\Omega)$. For simplicity of notations, we will denote $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\| = \|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{H^m(\Omega)}$ by $\|\cdot\|_m = \|\cdot\|_{m,\Omega}$.

The space $H^m(\Omega)$ is a Hilbert space with the norm induced by the natural inner product defined by

$$(\phi,\psi)_m := \sum_{0\le|\tilde{\gamma}|\le m}\int_\Omega D^{\tilde{\gamma}}\phi D^{\tilde{\gamma}}\psi \quad \forall \phi,\psi\in H^m(\Omega).$$

The space $H_0^m(\Omega)$ is also defined as the closure of $C_0^\infty(\Omega)$ in the H^m-norm. Moreover, the function space $H_0^1(\Omega)$ consists of the elements from $H^1(\Omega)$ which vanish on the boundary Γ of Ω, where the boundary values are to be interpreted in the sense of trace (the trace is a continuous (bounded) linear operator $\overline{\gamma} : H^1(\Omega) \to L^2(\Gamma)$ such that $\overline{\gamma}(u) = u|_\Gamma$).

For an integer $m \ge 0$, set $s = m + \tilde{\sigma}$, $0 < \tilde{\sigma} < 1$, and then $H^s = H^s(\Omega)$ denote the Sobolev spaces of fractional order with the norm defined by

$$\|u\|_{H^s} = \left(\|u\|^2_{H^m} + \sum_{|\bar{\alpha}|=m}\int\int_{\Omega\times\Omega}\frac{|D^{\bar{\alpha}}u(x) - D^{\bar{\alpha}}u(y)|^2}{|x-y|^{2+2\tilde{\sigma}}}dxdy\right)^{1/2},$$

and let $H^s = H^s(\Omega)$ denote the completion of $C^\infty(\overline{\Omega})$ with respect to this norm. For a given Banach space $\mathbf{B}$ and $1 \le p \le \infty$, we define the Bochner spaces $L^p(0,T;\mathbf{B})$ as

$$L^p(0,T;\mathbf{B}) = \Big\{\phi : (0,T) \to \mathbf{B} \;\Big|\; \phi(t)\in\mathbf{B} \text{ for almost all } t\in(0,T) \text{ and } \int_0^T \|\phi(t)\|^p_{\mathbf{B}}dt < \infty\Big\}$$

equipped with the norm

$$\begin{aligned}\|\phi\|_{L^p(0,T;\mathbf{B})} &:= \left(\int_0^T \|\phi(t)\|^p_{\mathbf{B}}\right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,\\ \|\phi\|_{L^\infty(0,T;\mathbf{B})} &:= \operatorname*{ess\ sup}_{t\in(0,T)}\|\phi(t)\|_{\mathbf{B}} < \infty, \quad p = \infty.\end{aligned}$$

For convenience, we write $\|\phi\|_{L^p(0,T;\mathbf{B})} = \|\phi\|_{L^p(\mathbf{B})}$. For having a complete idea on Sobolev spaces, one may refer to Adams [2], Adams and Fournier [3], Dautray and Lions [29], Grisvard [47] and Kellogg [56].

In the subsequent chapters, we shall use the following spaces and some preliminary materials. The space $H^1(0,T;\mathbf{B})$ consists of all measurable functions $\phi : (0,T) \to \mathbf{B}$ for which

$$\|\phi\|_{H^1(0,T;\mathbf{B})} := \left(\int_0^T \|\phi(t)\|^2_{\mathbf{B}}dt + \int_0^T \|\phi_t(t)\|^2_{\mathbf{B}}dt\right)^{1/2} < \infty.$$

Let $\|\cdot\|_{r,[0,T]}$ denote the norm on $H^r([0,T])$ and the space $H^{s,r}(\Omega_T) = L^2(0,T;H^s(\Omega)) \cap H^r(0,T;L^2(\Omega))$ be equipped with the norm

$$\|v\|_{s,r} = \left(\int_0^T \|v(\cdot,t)\|_s^2 dt + \int_\Omega \|v(x,\cdot)\|_{r,[0,T]}^2 dx \right)^{1/2},$$

where $\Omega_T = \Omega \times (0,T]$. We set

$$\begin{aligned} W(0,T) &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ X(0,T) &:= L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \end{aligned}$$

and define the space $\overline{X}(0,T)$ as

$$\overline{X}(0,T) := L^2(0,T;H^{1+s}(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \quad \text{for } 1/2 \le s \le 1.$$

The L^2-inner product on $L^2(\Omega_T)$ is defined by $(u,v)_{\Omega_T} = \int_{\Omega_T} uv dx dt \ \ \forall\, u,v \in L^2(\Omega_T)$.

Some inequalities. Now we shall recall some well known inequalities which will be used often in our chapters (cf. Hardy *et al.* [54]).

Young's inequality: If a, b are nonnegative integers, and $\epsilon > 0$ then

$$ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}.$$

An important consequence of the Young's inequality is the celebrated Hölder's inequality. The discrete and integral version of the Hölder's inequality are given as follows.

Hölder's inequality: The discrete analogue is stated as: Let $p > 1$ and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $a_i, b_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$,

$$\sum_{i=1}^n |a_i b_i| \le \left(\sum_{i=1}^n |a_i|^p \right)^{1/p} \left(\sum_{i=1}^n |b_i|^q \right)^{1/q}.$$

In particular, for $p = q = 2$, the above inequality is familiar as the Cauchy-Schwarz inequality in $\mathbb{R}^n$.

The integral analogue is stated as follows: Let $p, q \in [1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\phi, \psi : \Omega \to \mathbb{R}$ are Lebesgue mesurable. Then

$$\|\phi\psi\|_{L^1(\Omega)} \le \|\phi\|_{L^p(\Omega)} \|\psi\|_{L^q(\Omega)}.$$

In particular, for $p = q = 2$, the above inequality is known as the Cauchy-Schwarz inequality in integral form which will be used frequently.

Poincaré's inequality [16]: Let Ω be a bounded domain in $\mathbb{R}^n$. Then there exists a positive constant $C_P = C_P(\Omega)$ such that

$$\|\phi\| \le C_P \|\nabla\phi\| \quad \forall \phi \in H_0^1(\Omega).$$

An immediate consequence of the Poincaré's inequality is that, $\|\nabla(\cdot)\|$ defines a norm on $H_0^1(\Omega)$. We now state the Brouwer's fixed point theorem without proof. The proof of the theorem is provided in the article of Kellogg *et al.* [57].

Theorem 1.2.1 (Brouwer's fixed point theorem)**.** *Let X be a compact convex set in an n-dimensional Euclidean space. Then for any continuous function $f : X \to X$ there is a point $x_0 \in X$ such that $f(x_0) = x_0$.*

The simplest form of Brouwer's fixed point theorem in the plane is given as: Every continuous function from a closed disk to itself has at least one fixed point. Next, we state without proof, the continuous version of Gronwall's lemma. For a proof, see Rao [83].

Lemma 1.2.1 (Gronwall's lemma)**.** *Let $G(t)$ be a continuous function and $H(t)$ a non-negative continuous function on the interval $t \in [0, T]$. If a continuous function $F(t)$ has the property*

$$F(t) \leq G(t) + \int_0^t F(\zeta)H(\zeta)d\zeta \quad \textit{for } t \in [0, T],$$

then

$$F(t) \leq G(t) + \int_0^t G(\zeta)H(\zeta)exp\left(\int_\zeta^t H(\tilde{\zeta})d\tilde{\zeta}\right) d\zeta \quad \textit{for } t \in [0, T].$$

In particular, when $G(t) = C$, a nonnegative constant, we have

$$F(t) \leq C\, exp\left(\int_0^t H(\zeta)d\zeta\right) \quad \textit{for } t \in [0, T].$$

Existence, uniqueness and regularity results. The existence and uniqueness of the solution of parabolic problems for both linear and nonlinear models are investigated by numerous authors, see Douglas and Dupont [33], Girault and Raviart [45], Ladyženskaja *et al.* [63], Thomée [102], Zlàmal [105] and the respective references provided therein. The existence and uniqueness result for parabolic problems with measure data in time can be found in the work of Gong [46]. Due to presence of measure data in time the solution of such problems possess less regularity and hence the standard finite element error analysis trick cannot be applied directly. However, writing the weak formulation by transposition techniques (cf. Lions and Magenes [66]), the author of [46] has proved the existence and uniqueness of the solution to problem (1.4). The regularity of the solution of these problems introduced above (in Section 1.1) has been extensively studied, see Chrysafinos and Hou [24], Evans [38], Gong [46], Luskin and Rannacher [70] and Thomée [102] for convex domains and Babuška [6], Bacuta *et al.* [8, 9], Chatzipantelidis *et al.* [21, 22, 23], Grisvard [47, 48] and Kozlov *et al.* [60] for nonconvex domains.

1.3 Framework for the CFE Method

For the purpose of the CFE approximations of the model problems (1.1), (1.3) and (1.4) we begin by writing the problems in weak form. The weak form of problem (1.1) is given by: For each $t > 0$, find $u(t) \in H_0^1(\Omega)$ such that

$$
\begin{aligned}
(u_t(t), \varphi) + \mathcal{A}(u(t), \varphi) &= (f(t), \varphi) \quad \forall \varphi \in H_0^1(\Omega),\ t > 0, \\
u(0) &= u_0(x),
\end{aligned}
\tag{1.5}
$$

where $\mathcal{A}(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a bilinear form defined by

$$\mathcal{A}(v, w) = (\nabla v, \nabla w) \quad \forall v, w \in H_0^1(\Omega),$$

here ∇ denotes the spatial gradient. The bilinear form follows the following two properties: $\mathcal{A}(\cdot,\cdot)$ is bounded on $H_0^1(\Omega)$, i.e., $\exists\, \alpha_0 > 0$ such that

$$|\mathcal{A}(v, w)| \leq \alpha_0 \|v\|_1 \|w\|_1 \quad \forall v, w \in H_0^1(\Omega),$$

and $\mathcal{A}(\cdot,\cdot)$ is coercive on $H_0^1(\Omega)$, i.e., $\exists\, \beta_0 > 0$ such that

$$\mathcal{A}(v, v) \geq \beta_0 \|v\|_1^2 \quad \forall v \in H_0^1(\Omega).$$

The standard weak formulation of (1.3) may be stated as: Find $u : \bar{J} \to H_0^1(\Omega)$ such that

$$
\begin{gathered}
(u_t, \varphi) + (a(u)\nabla u, \nabla \varphi) = (f(u), \varphi) \quad \forall \varphi \in H_0^1(\Omega),\ t \in J, \\
u(0) = u_0(x).
\end{gathered}
\tag{1.6}
$$

Next, for parabolic problem (1.4) involving measure data, we define the weak formulation by transposition techniques [66] as follows: Find $u \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ such that

$$
\begin{aligned}
-(u, \varphi_t) + \mathcal{A}(u, \varphi)_{\Omega_T} &= \langle \mu, \varphi \rangle_{\Omega_T} + (u_0, \varphi(\cdot, 0)) \quad \forall \varphi \in W(0, T), \\
\text{with} \quad \varphi(\cdot, T) &= 0,
\end{aligned}
\tag{1.7}
$$

where $\mathcal{A}(u, \varphi)_{\Omega_T} = (\nabla u, \nabla \varphi)_{\Omega_T}$ and

$$\langle \mu, \varphi \rangle_{\Omega_T} = \int_{\overline{\Omega}_T} \varphi \, d\mu = \int_0^T \left(\int_\Omega \sigma(x, t) \varphi(x, t) dx \right) d\omega(t) \quad \forall \varphi \in C([0, T]; L^2(\Omega)).$$

Two-Scale CFE Discretization. In order to discretize the domain Ω, we introduce two-scale grid: The coarse-scale and the fine-scale grid (see e.g., Rech *et al.* [86]). The two-scale CFE discretization describes in the following steps:

Step 1 *(Two-scale grid refinement):* Let $\mathcal{T}_H = \{K_1, K_2, \cdots, K_n\}$ denote the coarse-scale grid that contains the *degrees of freedom*. Also, let $\mathcal{T}_h$ denote the fine-scale grid containing only *slave nodes* which adaptively resolves the boundary Γ. For a triangle $K \in \mathcal{T}_H$, h_K denotes the diameter of K. The index H denotes the largest diameter of the triangles in $\mathcal{T}_H$ and is defined by $H := \max\{h_K : K \in \mathcal{T}_H\}$.

We assume that $\mathcal{T}_H$ is an overlapping grid consisting of conforming shape regular triangles (cf. Ciarlet [25]) with maximal diameter H. As $\mathcal{T}_H$ is an overlapping grid, we have

$$\Omega \subset \bigcup_{K \in \mathcal{T}_H} K \quad \text{with} \quad \text{int}(K) \cap \Omega \neq \emptyset \quad \forall K \in \mathcal{T}_H,$$

where $\text{int}(K)$ denotes the interior of a triangle K. An adaptive procedure is used to refine the triangles in a certain neighborhood of Γ to resolve the boundary conditions. The width of this neighborhood is controlled by a parameter $\sigma_{\text{dist}} > 0$. An algorithm which marks two layers of triangles near the boundary Γ can be found in the article of Peterseim and Sauter [77] along with that of Rech [85]. For any $K \in \mathcal{T}_H$, the set of sons, denoted by $\text{sons}(K)$, is defined by

$$\text{sons}(K) := \{\tau \in \mathcal{T}_{H,h} \ : \ \tau \subset K\} \tag{1.8}$$

and its number is denoted by $n_K := \sharp\,\text{sons}(K)$, where $\mathcal{T}_{H,h}$ denotes the two-scale grid. To obtain τ (sons of K), we split each K into four triangles by joining the mid-points of the edges of K. This leads to a new conforming and shape regular grid in the vicinity of Γ, which is more refined than $\mathcal{T}_H$ and coincides with $\mathcal{T}_H$ in the interior of Ω. In the interior of the domain, at some distance from Γ, consider the submesh

$$\mathcal{T}_H^{\text{in}} := \{\tau \in \text{sons}(K) : K \in \mathcal{T}_H \setminus \mathcal{T}_\Gamma\} \subset \mathcal{T}_{H,h},$$

which is characterized by the coarse-scale parameter H, where $\mathcal{T}_\Gamma$ is a subset of $\mathcal{T}_H$ containing all near-boundary triangles. The two-scale mesh $\mathcal{T}_{H,h}$ is characterized by the fine-scale parameter h in the neighborhood of Γ, where $h := \min\{h_\tau : \tau \in \mathcal{T}_{H,h}\}$, $h \leq H$. So, the near-boundary triangles satisfy the following condition:

$$\text{dist}(K, \Gamma) \leq \sigma_{\text{dist}} h_K \quad \forall K \in \mathcal{T}_{H,h} \setminus \mathcal{T}_H^{\text{in}}. \tag{1.9}$$

More precisely, the stopping criterion of finer scale grid refinement must contain the condition (1.9). The two-scale CFE discretization for convex and nonconvex domains are depicted in Figures 1.1 and 1.2, respectively.

Step 2 *(Specify the degrees of freedom):* Let $\mathbf{G}(K)$ denote the set of vertices for any triangle K. We now define *free nodes* for locating the degrees of freedom and the *slave nodes*, where the function values are constrained. Let ϑ_H denote the set of all vertices in $\mathcal{T}_H$. The degrees of freedom corresponds to those vertices in the coarse mesh $\mathcal{T}_H$. More precisely, it depends on the inner mesh $\mathcal{T}_H^{\text{in}}$ with a proper distance to the boundary. Thus, the degrees of freedom is defined as

$$\vartheta_{\text{dof}} := \{x \in \mathbf{G}(K) : K \in \mathcal{T}_H^{\text{in}}\}.$$

All other nodes in $\mathcal{T}_{H,h}$ are called *slave nodes.* The values of a CFE function is determined by its values at the nodes $x \in \vartheta_{\text{dof}}$. The triangles and grid points are used for *adapting* the shape of the CFE functions to the Dirichlet boundary conditions. An algorithm for generating triangles and grid points can be found in the article of Hackbusch and Sauter [53] along with that of Peterseim and Sauter [77].

Step 3 *(Characterize the slave nodes):* Let $\vartheta_{H,h}$ denote the set of all vertices of the two-scale mesh $\mathcal{T}_{H,h}$. The set of slave nodes are defined by

$$\vartheta_{\text{slave}} := \vartheta_{H,h} \setminus \vartheta_{\text{dof}}.$$

For a slave nodes $x \in \vartheta_{\text{slave}}$, we determine a closest coarse grid triangle $\Delta_x \in \mathcal{T}_H^{\text{in}}$ and the closest point x^Γ on the boundary Γ:

$$x \to \Delta_x \in \mathcal{T}_H^{\text{in}} \quad \text{and} \quad x \to x^\Gamma \in \Gamma.$$

The mappings are illustrated in Figures 1.3a and 1.3b for convex and nonconvex domains, respectively.

Remark 1.3.1. *(a) For any $\tau \in sons(K)$, $K \in \mathcal{T}_H$, the local mesh size h_τ and the coarse mesh size H is related by $h_\tau \leq h_K \leq H$.*

(b) Let n_K denote the number of sub-triangles in $K \in \mathcal{T}_\Gamma$. If $n_K = 1$, then there is no subdivision of K. In this case, the two-scale grid $\mathcal{T}_{H,h}$ simply coincides with the coarse-scale grid $\mathcal{T}_H$ ($h = \mathcal{O}(H)$). The method is referred to as 'one-scale CFE method'. When $h \ll H$, this is called 'two-scale CFE method'. When the number n_K is very large, then the domain Ω is fully resolved by the grid $\mathcal{T}_{H,h}$.

Extrapolation operator. As the degrees of freedom of the CFE space are located at the inner nodes ϑ_{dof} on the inner mesh $\mathcal{T}_H^{\text{in}}$, the values at the slave nodes of the two-scale mesh $\mathcal{T}_{H,h}$ are determined by an extrapolation method.

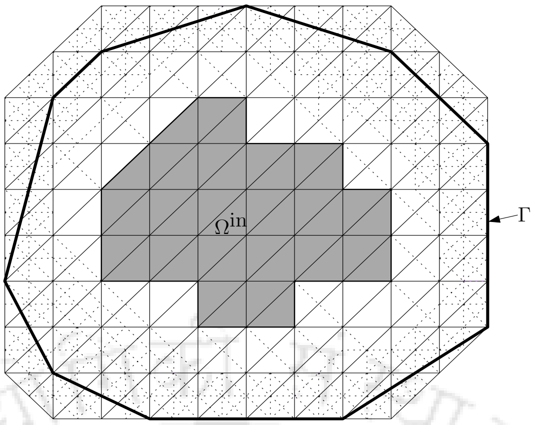


Figure 1.1: *CFE discretization for the two-scale grid $\mathcal{T}_{H,h}$ with smooth boundary Γ. The dark-shaded triangles form the inner triangulation $\mathcal{T}_H^{in}$ which contains the degrees of freedom. The near-boundary triangles are denoted by dotted lines which contain the slave nodes.*

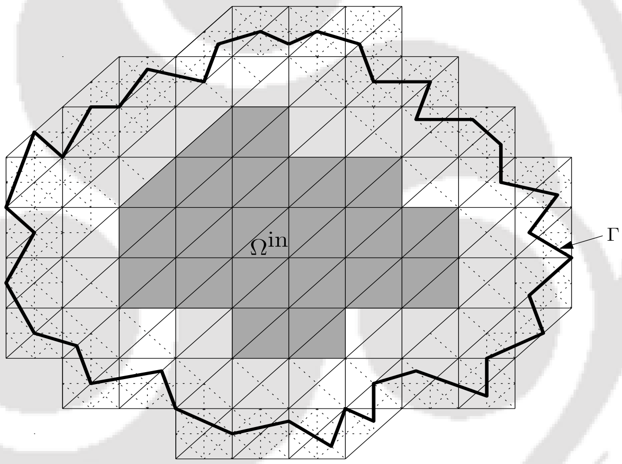


Figure 1.2: *CFE discretization for the two-scale grid $\mathcal{T}_{H,h}$ with the nonsmooth boundary Γ. The domain boundary Γ is shown by the solid line. The degrees of freedom is located corresponding to the dark-shaded region Ω^{in}.*

Let $\mathbf{\Phi} : \vartheta_{\text{dof}} \to \mathbb{R}$ denote a grid function. For any $K \in \mathcal{T}_H$, there exists a uniquely determined linear function $\Phi_K \in \mathbb{P}_1(\mathbb{R}^2)$ which interpolates $\mathbf{\Phi}$ in the vertices of K. Here, we use $\mathbb{P}_1(\mathbb{R}^2)$ to denote the space of linear polynomials on $\mathbb{R}^2$ of maximal degree 1. The values of the extension of $\mathbf{\Phi}$ at a slave node $x \in \vartheta_{\text{slave}}$ is defined by

$$(\mathcal{E}\mathbf{\Phi})_x := \Phi_{\Delta_x}(x) - \Phi_{\Delta_x}(x^\Gamma). \tag{1.10}$$

Note that $(\mathcal{E}\mathbf{\Phi})_x = 0$ for $x \in \Gamma$. The construction (1.10) ensures a sufficient approximation quality of the Dirichlet boundary condition. The relation (1.10) defines an

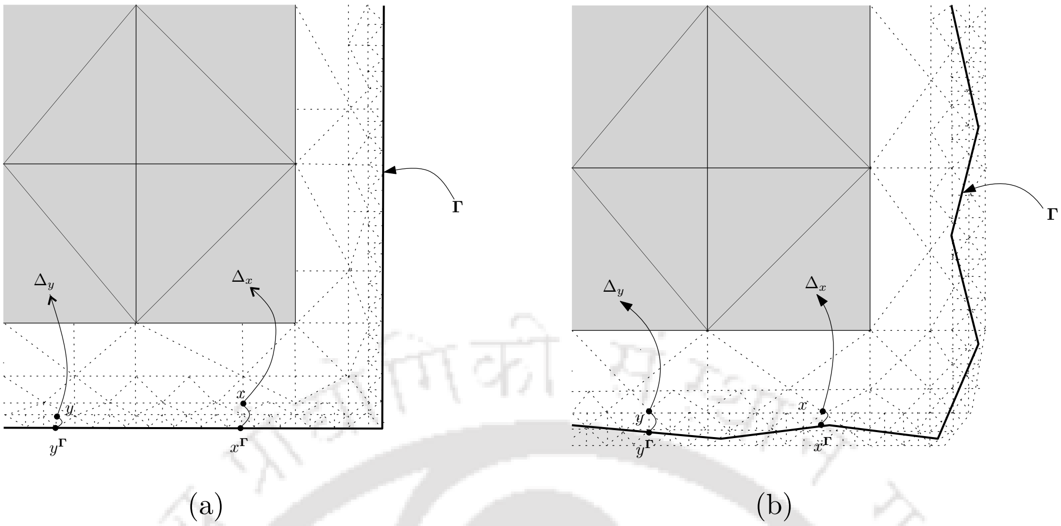


Figure 1.3: *(a) For slave node x (or y), the choice of closest inner simplex Δ_x (or Δ_y), and the closest boundary point x^Γ (or y^Γ), where the boundary Γ is smooth. (b) For slave nodes x (or y), the choice of closest inner simplex Δ_x (or Δ_y), and the closest boundary point x^Γ (or y^Γ), where the boundary Γ shown by the dark line is complicated.*

extrapolation operator $\mathcal{E} : \mathbb{R}^{\vartheta_{\text{dof}}} \to \mathbb{R}^{\vartheta_{H,h}}$ for the grid functions as follows:

$$(\mathcal{E}\mathbf{\Phi})_x := \begin{cases} \mathbf{\Phi}_x, & x \in \vartheta_{\text{dof}}, \\ \Phi_{\Delta_x}(x) - \Phi_{\Delta_x}(x^\Gamma), & x \in \vartheta_{\text{slave}} \ \wedge \ \text{dist}(x, \Delta_x) \leq \sigma_e h_{\Delta_x}, \\ 0, & \text{otherwise}, \end{cases} \tag{1.11}$$

where $\sigma_e > 0$ be some control parameter, which is essential to control the distance from an inner triangle Δ_x to a slave node x.

CFE space. With the above two-scale CFE discretizations for the domain Ω we are now ready to define the CFE space. Let S denote the continuous and piecewise linear finite element space on the mesh $\mathcal{T}_{H,h}$ be defined by

$$S := \{v \in C^0(\Omega_{H,h}) \ | \ v|_K \in \mathbb{P}_1 \ \ \forall K \in \mathcal{T}_{H,h}\},$$

where $\Omega_{H,h} := \text{int}\left(\bigcup_{K \in \mathcal{T}_{H,h}} K\right)$. Then the CFE space for the two-scale approximation of Dirichlet boundary conditions on the mesh $\mathcal{T}_{H,h}$ is defined by

$$S^{\text{CFE}} := \{v \in S \mid \exists \ \mathbf{\Phi} \in \mathbb{R}^{\vartheta_{\text{dof}}} \ \text{with} \ v(x) = (\mathcal{E}\mathbf{\Phi})_x \ \ \forall x \in \vartheta_{H,h}\}.$$

From the definition of S^{CFE} it is clear that $S^{\text{CFE}} \subset S$, where the values at the slave nodes are restricted by the extrapolation.

Remark 1.3.2. (*i*) *The dimension of S^{CFE} is determined by the number of nodes in ϑ_{dof}. Hence, the dimension of S^{CFE} may be much less than the dimension of S.*

(*ii*) *The triangles in the two-scale grid $\mathcal{T}_{H,h}$ might overlap the boundary Γ and thus the homogeneous boundary condition satisfied by the functions from S^{CFE} only approximately.*

Spatially semidiscrete CFE approximation. Now we define the spatially semidiscrete CFE approximations based on the weak formulations (1.5), (1.6) and (1.7). The spatially semidiscrete CFE approximation based on the weak form (1.5) is to find $u^{\mathrm{CFE}}(t) := u^{\mathrm{CFE}}(\cdot, t)$ with $u^{\mathrm{CFE}}(t) \in S^{\mathrm{CFE}}$ for $t \geq 0$ such that

$$\begin{aligned} (u_t^{\mathrm{CFE}}, \chi) + \mathcal{A}(u^{\mathrm{CFE}}, \chi) &= (f, \chi) \quad \forall \chi \in S^{\mathrm{CFE}},\ t > 0, \\ u^{\mathrm{CFE}}(0) &= u_0^{\mathrm{CFE}}, \end{aligned}$$

where u_0^{CFE} is some suitable approximation of u_0 in S^{CFE}. We now pose the spatially semidiscrete problem based on the variational form (1.6): Find $u^{\mathrm{CFE}} : \bar{J} \to S^{\mathrm{CFE}}$ such that

$$\begin{aligned} (u_t^{\mathrm{CFE}}, \chi) + (a(u^{\mathrm{CFE}})\nabla u^{\mathrm{CFE}}, \nabla \chi) &= (f(u^{\mathrm{CFE}}), \chi) \quad \forall \chi \in S^{\mathrm{CFE}},\ t \in J, \\ u^{\mathrm{CFE}}(0) &= u_0^{\mathrm{CFE}}. \end{aligned}$$

In addition, we define the spatially semidiscrete CFE approximation based on the weak formulation (1.7) as: Find $u^{\mathrm{CFE}}(t) \in L^2(0, T; S^{\mathrm{CFE}})$ such that

$$-(u^{\mathrm{CFE}}, v_t^{\mathrm{CFE}})_{\Omega_T} + \mathcal{A}(u^{\mathrm{CFE}}, v^{\mathrm{CFE}})_{\Omega_T} = \langle \mu, v^{\mathrm{CFE}} \rangle_{\Omega_T} + (u_0^{\mathrm{CFE}}, v^{\mathrm{CFE}}(\cdot, 0))$$

$\forall v^{\mathrm{CFE}} \in H^1(0, T; S^{\mathrm{CFE}})$, with $v^{\mathrm{CFE}}(\cdot, T) = 0$ and

$$\langle \mu, v^{\mathrm{CFE}} \rangle_{\Omega_T} = \int_{\overline{\Omega}_T} v^{\mathrm{CFE}} d\mu = \int_0^T \left(\int_\Omega \sigma(x,t) v^{\mathrm{CFE}}(x) dx \right) d\omega(t) \quad \forall v^{\mathrm{CFE}} \in H^1(0, T; S^{\mathrm{CFE}}).$$

Now we shall turn to introduce some notations for the fully discrete schemes. As a fully discrete approximation we consider the *backward Euler* and the *Crank-Nicolson* schemes for discretization with respect to the time variable. Let $0 = t_0 < t_1 < \cdots < t_n < \cdots$ (n is a nonnegative integer) be a partition of the positive time axis and let k_i be the variable time step defined by $k_i = t_i - t_{i-1}$. For a bounded time interval $\bar{J} = [0, T]$, we consider the grid points along time axis

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T,$$

and the corresponding time interval partitioning as

$$\bar{J} = \{0\} \cup J_1 \cup J_2 \cup \cdots \cup J_N,$$

with each subintervals $J_i = (t_{i-1}, t_i]$ of size k_i. For the constant time step, in each time interval we have $k_i = k$, i.e., the size of each subintervals $(t_{i-1}, t_i]$ is k. The fully discrete approximation of the exact solution $u(t)$ at $t = t_n = nk$ in S^{CFE} is denoted by U^n. By u^n and f^n we mean that $u^n = u(\cdot, t_n) = u(t_n)$ and $f^n = f(\cdot, t_n) = f(t_n)$ for $t = t_n$. Throughout this dissertation, we shall use the backward Euler quotient

$$\bar{\partial} U^n := (U^n - U^{n-1})/k \qquad \text{and} \qquad \bar{\partial}_n U^n := (U^n - U^{n-1})/k_n$$

for constant and variable time steps, respectively. Moreover, for the Crank-Nicolson scheme the solution U^n is approximated at $t = t_{n-\frac{1}{2}} = (n - \frac{1}{2})k$ with $\bar{U}^n := (U^n + U^{n-1})/2$.

1.4 Background and Motivation

This section presents a brief survey of the relevant literature concerning two-scale CFE approximations for PDEs and their numerical solutions. Parabolic problems for linear and nonlinear models and parabolic problems with measure data in time for both convex and nonconvex domains appear in a wide variety of applications in science and engineering. There are several numerical methods in the literature designed for numerical approximation of these kind of problems. In practice, a discretization procedure such as finite difference method (FDM) and finite element method (FEM) are adapted. We first describe a brief account of the literature concerning the FDM for solving parabolic problems in convex and nonconvex domains.

Finite difference methods: There has been several considerable research to solve parabolic PDEs using FDM, see [1, 5, 28, 73, 81, 89, 96, 101] and references therein. Astrakhantsev [5] has constructed the finite difference schemes on a uniform mesh for solving parabolic equations in a two-dimensional region, where the region Ω is bounded and simple connected. The authors of [73] have studied the monotone FDM with an adaptive grid refinement technique to produce a PDE discretization which is applied to a broad class of equations, in curved or unbounded domains inclusive of free boundaries. They have considered the grid refinement to be flexible and adaptive. The application of the generalized FDM for solving the advection-diffusion equation has been investigated by Prieto *et al.* [81]. The authors have formulated an explicit method to study the convergence of the method and examined the truncation error over irregular grids. In [89], Sapa has approximated the classical solutions of nonlinear second-order partial differential functional equations of parabolic type with Neumann's condition by solutions of associated explicit difference functional equations, where the functional dependence is of the Volterra type. Sinsoysal and Coruhlu [96] have studied the finite differences scheme for finding a numerical solution of the parabolic equation with double nonlinearity. For this purpose, they have introduced a special auxiliary problem which have some advantages over the main problem. Using the advantages of the proposed auxiliary problem, they have derived the convergence of the numerical solution to the exact solution. The numerical solution to linear PDEs of parabolic type by means of FDM has been studied by Thomée in [101]. For further detailed discussion on FDM for parabolic problems, we refer to [1, 28] and the references quoted therein.

Finite element methods: In recent years, FEM has emerged as a very powerful numerical technique for finding approximate solution of PDEs over a given domain. For the decades, several authors have been investigated the error analysis and the convergence properties of FEM for different class of problems on the domains with smooth boundary, or convex polygonal domains and the domains with complicated boundary, or nonconvex polygonal domains.

There are wide range of articles available in the literature regarding *a priori* error estimates for parabolic IBVPs for convex and nonconvex domains. In [102], Thomée has considered the simplest Galerkin FEM corresponding to the IBVPs for the linear and nonlinear parabolic equations. The discrete methods are based on associated weak, or variational formulations of the problems. The author has demonstrated *a priori* error estimates in energy and mean square norms for these model problems. The error estimates are derived first for the spatially semidiscrete problem resulting from discretization in the spatial variables only. Then error estimates are also derived for the most commonly used fully discrete schemes as obtained by discretization in both space and time, such as the backward Euler and Crank-Nicolson methods. Moreover, the author has studied the nonsmooth data error estimates for homogeneous parabolic equation for positive time, where the given initial data is assumed to be in $L^2(\Omega)$. The error estimates by using the theory of analytic semigroups generated by the elliptic operator and its discrete analogue (cf. [37, 74, 98]) have also been studied. Bramble *et al.* [15] have analyzed the multi-step backward difference methods for the discretization of parabolic problems with smooth and nonsmooth data. The maximum norm stability and error analysis for parabolic finite element equations in convex domain have been investigated by Schatz *et al.* [93]. Subsequently, in [80], Price and Varga have investigated the numerical approximation of the solution of linear diffusion-convection equations for one spatial variable and derived error bounds for the semidiscrete Galerkin approximations of such parabolic problems.

A variety of Galerkin methods have been studied by Douglas and Dupont [34] for the parabolic equation subject to the nonlinear boundary condition with the usual initial condtion. The authors have obtained an optimal order error estimates in the L^2 and H^1-norms. In [35], the authors have estimated the errors for a class of Galerkin methods for a quasilinear parabolic equation. They have proved an asymptotic error estimates of optimal order. Luskin [68] has analyzed a Galerkin method for nonlinear parabolic equations with nonlinear boundary conditions, where an extrapolation in time is used to yield a system of linear algebraic equations to be solved at each time level. The development of the theory of finite elements for parabolic problems can also be found in [39, 40, 104] for linear models and [31, 32, 82] for nonlinear models.

The numerical treatment of elliptic and parabolic equations by the standard FEM involving measure data have been developed by numerous authors. The summability results for the gradients of solution of nonlinear parabolic equations having measure data with homogeneous Cauchy-Dirichlet boundary conditions are thoroughly studied by Boccardo *et al.* [13]. The existence of solution for equations of elliptic and parabolic type with bounded measure in a bounded open set Ω has been extensively studied, see Boccardo and Gallouët [14]. The first contribution for the finite element approximations of second order elliptic problem involving measure data in convex domains with homogeneous Dirichlet condition on the boundary is given by Casas [18]. In this article, the author has considered the data to be real and regular Borel measures. Using a quasi-uniform mesh of finite elements and polynomials of degree ≤ 1, the author has shown that in two-dimensions the convergence is of order $\mathcal{O}(h)$ in the L^2-norm and in three dimensions of order $\mathcal{O}(h^{1/2})$. The convergence analysis of the linear finite element approximate solution of a diffusion equation to a weak solution, with weak regularity assumptions on the data has been investigated by Gallouët and Herbin [42]. Recently, Gong [46] has investigated *a priori* error estimates for the finite element approximations of parabolic equations with measure data in an open bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3). The author has considered parabolic problem for both measure data in time and space, respectively. For both cases standard piecewise linear and continuous finite elements is used for the space discretization, while the backward Euler method is then used for time discretization and *a priori* error estimates for the semidiscrete and fully discrete problems are derived in the $L^2(L^2)$-norm. Due to the presence of measure data the solution of these kind of problems exhibit low regularity. For the purpose of spatially semidiscrete and fully discrete error analysis the author has used the duality argument by introducing a backward in time parabolic problems.

Complicated domains: Up to now, all literature presented above are about the convex domains. We shall now give some relevant literature concerning finite element approximations of elliptic and parabolic problems in nonconvex polygonal domains. The most important article in this direction was presented by Babuška [6]. It is known fact that the rate of convergence (ROC) of the FEM is greatly influenced by the existence of corners on the boundary. While solving nonhomogeneous problem $Lu = f$ with homogeneous boundary conditions, where L is a second order elliptic differential operator, the ROC for FEM strongly depends on the smoothness of the domain. Due to the presence of the corners on the domain boundary the solution does not have full regularity, and this causes a reduction in the ROC. In this article, the author has dealt with the problem of proper refinement in the FEM around the corners that leads to the ROC which is same as it would be on the domain with smooth boundary. Bacuta *et al.* [8] have studied the regularity estimates for elliptic boundary value problems (BVPs) in the framework of Besov spaces. The analysis is based on new interpolation results and multilevel representations of norms on Sobolev and Besov spaces. The first significant contribution towards the finite element error analysis of parabolic equations in nonconvex polygonal domains is due to Chatzipantelidis *et al.* [22]. They have addressed the IBVP for the heat equation of linear models with homogeneous Dirichlet boundary conditions and considered the semidiscrete and fully discrete approximations of its solution by piecewise linear finite elements in space. The semidiscrete error estimate are based on the following decomposition of the main error $\bar{e} := u_h - u$ as

$$\bar{e}(t) = (u_h(t) - R_h u(t)) + (R_h u(t) - u(t)),$$

where $u_h(t) = u_h(\cdot, t)$ and $u(t)$ denote the semidiscrete finite element solution and the exact solution of the parabolic problem, respectively. Here, $R_h : H_0^1(\Omega) \to S_h$ is the Ritz projection as introduced by Wheeler in [104] and is defined by

$$\mathcal{A}(R_h v - v, \chi) = 0 \quad \forall \chi \in S_h, \; v \in H_0^1(\Omega),$$

where $S_h \subset H_0^1(\Omega)$ denotes the piecewise linear finite element space based on the triangulations of Ω. A special feature in a nonconvex polygonal domain is the presence of singularities in the solutions generated by the corners even when the forcing term is smooth. These cause a reduction of the convergence rate in the FEM. In [22], the authors have demonstrated that by a systematical mesh refinements toward the nonconvex corner (cf. [84]) the optimal order $\mathcal{O}(h)$ and $\mathcal{O}(h^2)$ convergence is obtained in the H^1 and L^2 norms, respectively. Recently, Gao *et al.* [43] have studied the stability and convergence of fully discrete Galerkin FEM for the nonlinear thermistor equations in a nonconvex polygon. Here, they have established the unconditional stability and optimal error estimates of a linearized backward Euler Galerkin FEM for the time-dependent nonlinear thermistor equations in a two-dimensional nonconvex polygon. We refer to Li and Zhang [64], Chatzipantelidis *et al.* [21, 23], Bacuta *et al.* [9], Schatz and Wahlbin [94] and references therein for further works in this direction.

Composite finite elements (CFEs): We now define a new class of finite elements, so-called CFEs, for the discretization of parabolic problems in a two-dimensional convex as well as nonconvex polygonal domains. The basic idea of the CFE procedure is to work with fewer degrees of freedom by allowing finite element mesh to resolve the domain boundaries and to preserve the asymptotic order of convergence on coarse-scale mesh. In contrast to standard finite elements, CFE method uses the minimal dimension of the approximation space. The dimension of the CFE space is independent of the domain geometry and this is especially advantageous for problems on complex domains. These new class of finite elements can be interpreted as a generalization of standard finite elements by allowing the approximation of the domain boundaries in a flexible adaptive manner.

It is for the readers' ease in understanding the corresponding results obtained in the present thesis work, we give a brief account of the literature concerning CFEs. The idea of CFEs was first introduced by Hackbusch and Sauter [50] to define finite element spaces which have the asymptotic approximation property with possibly less number of unknowns and is independent of the shape of the domain. They have described how CFEs can be used for Galerkin discretization in combination with standard multi-grid methods. Later, in [51, 52], Hackbusch and Sauter have used CFEs for an approximation of elliptic BVP. The CFEs can be used for coarsening finite element spaces and the coarse space dimension is independent of the boundary conditions. In [52], they have presented a strategy for coarsening finite element spaces independent of the shape of the object and used multi-grid methods for solving PDEs on domains with complex boundary. In this article, they have focused only on the theoretical analysis to study the basic approximation results. Whereas in [51], they have concentrated on the implementation details of the theoretical aspects and presented numerical experiments. In [53], Hackbusch and Sauter have considered CFEs for the discretization of PDEs on complicated domains containing small micro-structures. They have used CFE method in conjunction with multi-grid methods to obtain very coarse-scale discretization with extremely few degrees of freedom. Recently, Rech *et al.* [86] have proposed the CFE method for the discretization of elliptic problems with Dirichlet boundary conditions in both convex and nonconvex domains. For the proposed FEM they have proved an optimal order approximation (up to logarithmic terms), and the error estimates are also valid for the cases when the exact solution has a reduced regularity due to re-entering corners of the domain boundary. Their work is based on the two-scale CFE discretization:

- The coarse-scale H describes the approximation of the solution in the interior of the domain at a proper distance to the boundary, and
- the fine-scale h describes the local mesh size which is used for the approximation of Dirichlet boundary conditions.

The boundary (conditions) are resolved in an adaptive way by refining the triangles in a certain neighborhood of Γ. For this purpose, the authors of [86] have adopted the following procedures (algorithms):

(i) Procedure mark_near_boundary_triangles,

(ii) procedure adapt_boundary (successively refines the near-boundary triangles),

(iii) procedure green_closure (eliminates all hanging nodes in the actual triangulation $\mathcal{T}_{H,h}$).

In [92], Sauter and Warnke have studied the mathematical analysis of the CFE approximation to elliptic BVPs with discontinuous coefficients, where the method preserves the continuity across the interface in an approximate way. Recently, Antonietti *et al.* [4] have introduced the *hp*-version discontinuous Galerkin CFE method for the discretization of second-order elliptic PDEs. A relatively complete theory on the composite mini element (CME) for the mixed discretization of the Stokes equations on two and three-dimensional domains, which may contain a huge number of geometric details, has been developed by Peterseim and Sauter [77]. The main idea is to adapt the shape of the finite element functions instead of resolving the geometric details by the finite element mesh. This CMEs belongs to the class of CFEs which have been introduced to Poisson-type problems and to problems in linear elasticity (cf. [52, 86]). More precisely, the authors of [77] have generalized CFEs to the Stokes problem. Schwen *et al.* [95] have presented the CFE method for 3D elasticity with discontinuous coefficients. Rather than using complicated and computationally expensive tetrahedral meshing, they have used a uniform hexahedral grid in order to construct the CFE basis functions. Later, Preusser *et al.* [79] have been developed the CFE approach for scalar and vector-valued elliptic BVPs with discontinuous coefficients across geometrically complicated interfaces. The construction of composite basis functions is based on a suitable local interpolation on the space of admissible functions. A more detailed discussions on coarsening finite element spaces can be found in the literature, see Bank and Xu [10, 11], Chan and Smith

[20], Kornhuber and Yserentant [58], Ruge and Stüben [87] and the references quoted therein.

In the present work, we will introduce the two-scale CFE method for parabolic time dependent problems in both convex and nonconvex polygonal domains. The numerical treatment by means of CFEs have been received utmost attention in the recent years. Though various studies have been conducted on the CFE error analysis for elliptic BVPs on convex and nonconvex domains, yet the CFE method for parabolic problems on convex and nonconvex domains are needs to be explored. This is so because the particular field has not been illumined till date as per the knowledge of the present author. In standard finite element discretizations, the computational effort depends on the number of elements which are necessary to resolve the domain. Therefore, a certain adaptivity has to be employed near the boundary for classical FEM to converge at an optimal rate. As a consequence, the dimension of the solution space can be very large for the domains with complicated structures. However, it is convenient to use CFEs which are based on the idea of incorporating the geometric complexity of physical domains into the shape of basis functions rather than into the finite element mesh, see [50, 51, 53]. The coarse mesh in CFE discretization does not need to be aligned with the boundary, whereas this is necessary for the classical FEM. Further, the underlying discretizations use unstructured grids to refine the computational grid in the vicinity of the boundaries until a desired approximation quality is reached. In contrast to standard finite elements, CFE method uses the minimal dimension of the approximation space. These CFEs can be used for coarsening finite element spaces and the coarse space dimension is independent of the boundary conditions.

Since elliptic error estimates play a crucial role in the error analysis for parabolic problems, therefore an attempt has been made in this thesis to generalize two-scale CFEs for elliptic problems to parabolic problems in convex and nonconvex polygonal domains. The two-scale CFE discretization uses two mesh parameters, the coarse-scale mesh parameter H and the fine-scale mesh parameter h. The domain Ω is overlapped by a uniform coarse-scale mesh $\mathcal{T}_H$ with mesh width H which is independent of the geometry. The resulting two-scale mesh $\mathcal{T}_{H,h}$ is obtained by adaptively refined the mesh $\mathcal{T}_H$ towards the boundary Γ. More precisely, the degrees of freedom corresponds to the unrefined triangles with diameter H in the interior of Ω. The CFE space coincides with the classical, piecewise linear finite element functions on the coarse inner triangles, and on the remaining vertices (slave nodes) of the two-scale mesh $\mathcal{T}_{H,h}$, the CFE values are determined via a simple extrapolation process introduced before. Thus, the dimension of the CFE space do not increase by the triangles and grid points. To begin with, we first study *a priori* error estimates for the spatially semidiscrete and fully discrete CFE approximation of the linear parabolic problem of the form (1.1), where Ω is a bounded convex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary Γ. For the purpose of spatially semidiscrete CFE error analysis, we decompose the total error as

$$\begin{aligned} e(t) &:= u^{\text{CFE}}(t) - u(t) \\ &= \{u^{\text{CFE}}(t) - R^{\text{CFE}}u(t)\} + \{R^{\text{CFE}}u(t) - u(t)\} \\ &:= \theta(t) + \rho(t), \end{aligned}$$

where R^{CFE} is the elliptic or Ritz projection defined in (2.8). Thus, the semidiscretization CFE error bounds on the main error e in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms are obtained by using the error bounds of ρ and ρ_t (see Lemma 2.4.1 and equations (2.13) and (2.14)), and optimal order convergence (up to logarithmic terms) are established. In order to estimate ρ in $L^\infty(H^1)$-norm, we consider two cases depending on the two-scale grid:

- For $K \in \mathcal{T}_H^{\text{in}}$, the estimate of ρ follows from the standard interpolation estimate, since on the inner triangulations $\mathcal{T}_H^{\text{in}}$ the CFE space coincides with the classical piecewise linear finite element functions.

- For $K \in \mathcal{T}_\Gamma$, the estimate follows from the local estimate by introducing the Lagrange interpolation operator on τ, $\tau \in \text{sons}(K)$.

It is noteworthy that for 'one-scale CFE method', i.e., for the case $h = \mathcal{O}(H)$, the result coincides with the standard FEM. On the other hand, the fully discrete error analysis is based on the backward Euler and Crank-Nicolson method. We again decompose the main error $e = U^n - u(t_n)$ into two parts, $\theta^n = U^n - R^{\text{CFE}}u(t_n)$ and $\rho^n = R^{\text{CFE}}u(t_n) - u(t_n)$. In the fully discrete error estimates we analyze the bounds of several errors namely, backward Euler error analysis for constant time step, backward Euler error analysis for variable time step and error analysis for the Crank-Nicolson method. Our next aim is to study the spatially semidiscrete and fully discrete error estimates for the solution of homogeneous parabolic equation (1.2) for nonsmooth initial data in a convex polygonal domain. Optimal error estimates (up to logarithmic terms) for positive time is proved in the $L^\infty(L^2)$-norm. The eigenfunctions expansion related to the elliptic operator and the rational approximations for exponential are the key technical tools used in nonsmooth data error analysis. The space discretization error is based on some auxiliary results which estimate the bounds for the total error e in terms of ρ (see Lemmas 3.2.1 and 3.2.2). The backward Euler method is utilized for the error estimation in the fully discrete scheme.

Our succeeding objective is to study the CFE error analysis for linear parabolic problem (1.1) in a nonconvex polygonal domain Ω, where both smooth and nonsmooth initial data cases are considered separately. The existence and uniqueness of the solution of spatially semidiscrete problem relies on the construction of CFE nodal basis functions, which is forged by using the extrapolation operator defined in (1.11). It is a known fact that for convex domains the exact solution u of (1.1) belongs to $H^2(\Omega) \cap H_0^1(\Omega)$. However, it is very likely to have a lower regularity for the exact solution in nonconvex polygonal domains (especially for the complicated polygonal boundary Γ) due to the existence of re-entrant corners on the boundary. Therefore, for a nonconvex domain $u \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$, $s \in [\frac{1}{2}, 1]$ (cf. Rech *et al.* [86]). In order to derive the estimates for $\nabla\rho$ we introduce an operator $\mathcal{L}$ that maps u to the error $(u - R^{\text{CFE}}u)$ is defined by $\mathcal{L}u := u - R^{\text{CFE}}u$ (see Lemma 4.2.1). Observe that the operator $\mathcal{L}$ is linear as the projection is a linear operation in Hilbert spaces. For the purpose of estimation of ρ we use the duality argument. Several error estimates are established for the linear parabolic problem (1.1) in nonconvex domains namely, spatially semidiscrete errors in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms, fully discretization errors for the backward Euler method with constant and variable time steps for smooth initial data, and the spatially semidiscrete and fully discrete backward Euler error estimates for nonsmooth initial data.

Proceeding further we are aimed to consider the CFE approximation to a model nonlinear parabolic equation (1.3) in nonconvex polygonal domains. We show error estimates for the spatially semidiscrete problem as well as the fully discrete backward Euler method. By introducing the elliptic projection $\tilde{u}^{\text{CFE}} = \tilde{u}^{\text{CFE}}(t)$ of the exact solution u of (1.3) in S^{CFE} defined in (5.5), we split the term as $e(t) = u^{\text{CFE}}(t) - u(t) = \tilde{\theta}(t) + \tilde{\rho}(t)$, where $\tilde{\theta}(t) = u^{\text{CFE}}(t) - \tilde{u}^{\text{CFE}}(t)$ and $\tilde{\rho}(t) = \tilde{u}^{\text{CFE}}(t) - u(t)$. Therefore, *a priori* error bound in the $L^\infty(L^2)$-norm for the semidiscrete problem comprises of the bounds for temporal error and its derivative (see Lemmas 5.2.2 and 5.2.3). The backward Euler method is used for the discretization with respect to time variable. An use of Brouwer's fixed point theorem guarantees the existence of the solution of fully discrete approximation of the spatially semidiscrete problem. The error estimates for the backward Euler scheme and its linearized modification are derived and analyzed.

Finally, we now concentrate on the error estimates for the two-scale CFE approximations of parabolic equation (1.4) with measure data in time for both convex and nonconvex domains. The solution of (1.4) exhibit low regularities mainly due to two reasons: One, the existence of measure data and another, the presence of singularities in the solution caused by the re-entrant corner. The theoretical and numerical analysis for parabolic equations with measure data in time is not straightforward due to the presence of measure data. The key ingredients in our error analysis are estimates for the associated elliptic projection in the framework of CFE method (see Lemmas 6.2.2 and 6.2.3) and the duality trick. The spatially semidiscrete error estimates in the $L^2(L^2)$-norm for both convex and nonconvex domains rely on bound for the errors of dual problem in the $L^\infty(L^2)$-norm in convex and nonconvex domains, respectively (see Lemmas 6.3.1 and 6.4.1). As a fully discrete approximation we consider the backward Euler scheme for discretization with respect to time variable. The fully discrete error estimates for the backward Euler scheme in convex and nonconvex domains are obtained by using the stability estimate for the solution of fully discrete CFE approximation.

1.5 General Outline of the Thesis

The present dissertation comprises of eight chapters and is organized as follows.

Chapter 1 introduces the problems and it contains the basic notations and preliminary materials to be used throughout the thesis. The CFE discretizations via introducing a two-scale grid and the construction of the CFE space via a simple extrapolation method are presented. In addition, this chapter also refers to the motivation behind the present study.

Chapter 2 is devoted to the *a priori* error analysis for the spatially semidiscrete and fully discrete CFE approximations of the linear parabolic problem (1.1) in a convex domain with smooth initial data. The error estimates in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms are established for both continuous time semidiscrete and fully discrete CFE methods. The fully discrete CFE methods are based on the backward Euler and Crank-Nicolson approximations. The backward Euler scheme is considered for both constant and variable time steps.

Chapter 3 addresses the nonsmooth data error estimates for both the spatially semidiscrete and fully discrete CFE approximations of the homogeneous linear parabolic equation (1.2) in a convex domain. We have demonstrated an analogous behaviour for the CFE solution which implies that optimal order convergence (up to logarithmic terms) takes place in both spatially semidiscrete and fully discrete methods for positive time even for nonsmooth initial data.

Chapter 4 presents *a priori* error analysis of the spatially semidiscrete and fully discrete CFE approximations of the linear parabolic problem (1.1) in a nonconvex domain. Both smooth and nonsmooth initial data cases are considered. The convergence properties for both spatially semidiscrete and fully discrete CFE methods in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms are derived. The backward Euler method is employed for the fully discrete scheme for both smooth and nonsmooth initial data.

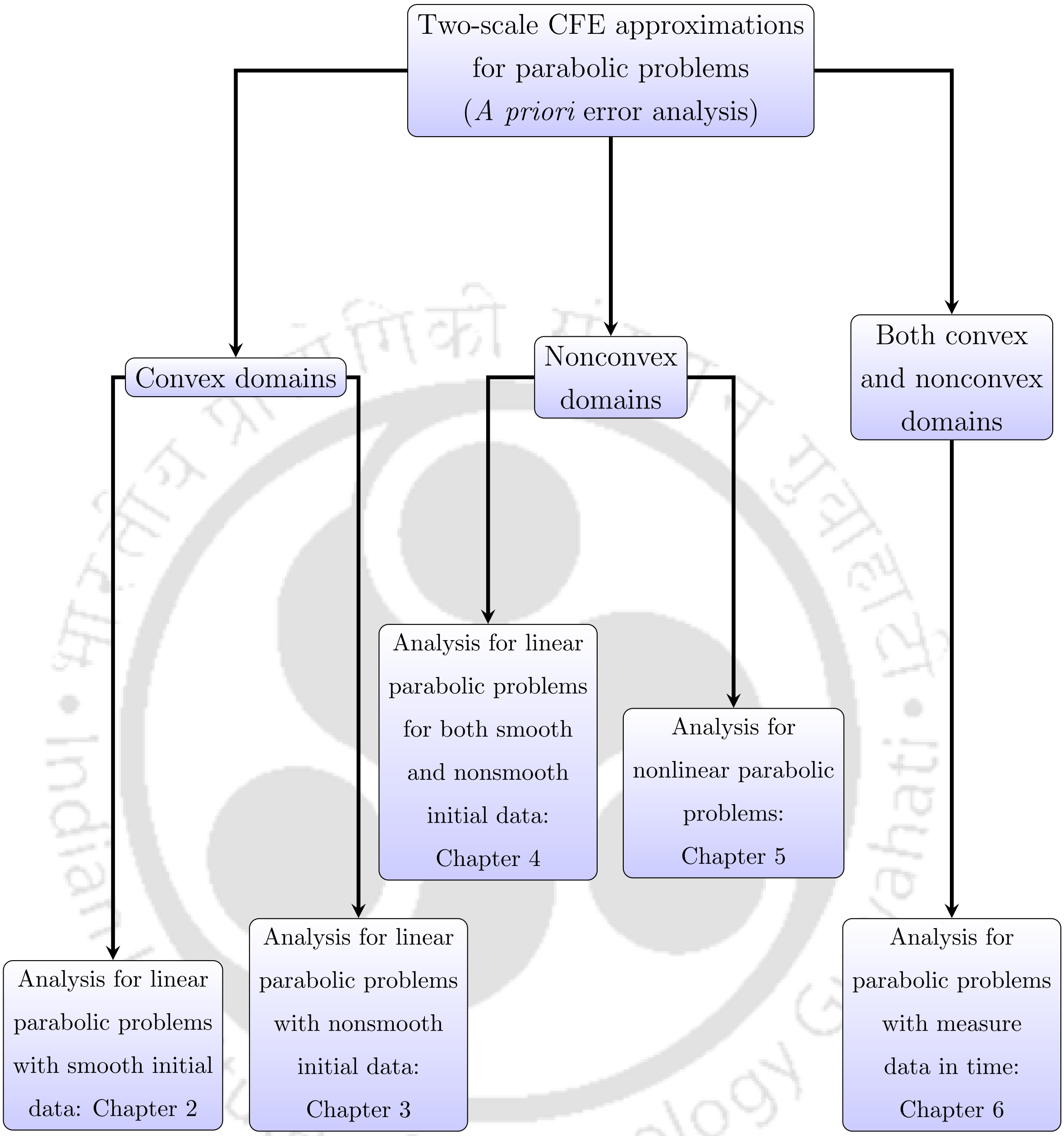
In Chapter 5, we address the CFE approximation for nonlinear parabolic problem (1.3) in nonconvex domains. Both spatially semidiscrete and fully discrete CFE methods are analyzed. The fully discrete backward Euler method (nonlinearized form) has a disadvantage that at each time level it is needed to solve a nonlinear system of algebraic equations because of the presence of two nonlinear terms $a(U^n)$ and $f(U^n)$. Thus, to overcome this difficulty, a linearized modification of the fully discrete backward Euler method has been considered and analyzed. A convergence of order $\mathcal{O}(H^{2s}\widetilde{\mathrm{Log}}^{s}(H/h) + k)$, $1/2 \le s \le 1$ is established for the fully discrete method.

In Chapter 6, we study *a priori* error analysis for parabolic problems (1.4) with measure data in time for both convex and nonconvex domains. The spatially semidiscrete and fully discrete error estimates in the $L^2(L^2)$-norm are demonstrated for both convex and nonconvex domains. Moreover, the fully discretization error analysis, based on the backward Euler method is done for both the cases. We derive error estimate of order $\mathcal{O}(H\widetilde{\mathrm{Log}}^{1/2}(H/h))$ for convex domains and $\mathcal{O}(H^s\widetilde{\mathrm{Log}}^{s/2}(H/h))$, $1/2 \le s \le 1$ for nonconvex domains with respect to the space discretization. An order $\mathcal{O}(k^{1/2})$ is obtained for the time discretization.

In Chapter 7, we perform numerical experiments for a two-dimensional test problem to validate our theoretical results derived in Chapters 2 – 6. For convex domains, a comparison between the CFE method and the standard FEM in terms of number of degrees of freedom ($\sharp$ dof) is presented.

Finally, Chapter 8 discusses the critical evaluation of the results highlighting the contributions made by the thesis. It also provides information for the scope of future investigations.

For clarity of presentation we have repeatedly mentioned the relevant preliminary materials at the beginning of each chapter. The pictorial representation of the thesis is shown as follows.


Two-scale CFE approximations for parabolic problems (*A priori* error analysis)
Convex domains
Nonconvex domains
Both convex and nonconvex domains
Analysis for linear parabolic problems for both smooth and nonsmooth initial data: Chapter 4
Analysis for nonlinear parabolic problems: Chapter 5
Analysis for linear parabolic problems with smooth initial data: Chapter 2
Analysis for linear parabolic problems with nonsmooth initial data: Chapter 3
Analysis for parabolic problems with measure data in time: Chapter 6

2

Two-Scale CFE Method for Parabolic Problems in Convex Polygonal Domains with Smooth Initial Data

In this chapter, we derive *a priori* error estimates for the spatially semidiscrete and fully discrete CFE approximations of the problem (1.1) in a convex polygonal domain with smooth initial data. The error analysis is based on the error splitting technique. The estimates of the elliptic or Ritz projection in the framework of CFE method play crucial role in deriving the error estimates. The convergence properties in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms are demonstrated for both semidiscrete and fully discrete CFE methods.

2.1 Introduction

To begin with, we first recall the linear parabolic problem of the form

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } \Omega,\ t > 0, \\ u &= 0 \quad \text{on } \Gamma,\ t > 0, \\ u(\cdot, 0) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded convex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary Γ. Here u and f are functions of x and t, u_t denotes $\partial u/\partial t$, and the Laplacian is denoted by $\Delta = \sum_{j=1}^{2} \partial^2/\partial x_j^2$. The initial data u_0 is a function of x only and is assumed to be smooth.

As a first step towards CFE approximation of (2.1), we recall the bilinear form $\mathcal{A}(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{A}(v, w) = (\nabla v, \nabla w) \quad \forall v, w \in H_0^1(\Omega).$$

The bilinear form $\mathcal{A}(\cdot,\cdot)$ is bounded and coercive on $H_0^1(\Omega)$, i.e., $\exists\, \alpha_0,\ \beta_0 > 0$ such that

$$|\mathcal{A}(v, w)| \leq \alpha_0 \|v\|_1 \|w\|_1 \quad \forall v, w \in H_0^1(\Omega),$$

and

$$\mathcal{A}(v,v) \geq \beta_0 \|v\|_1^2 \quad \forall v \in H_0^1(\Omega).$$

Then the weak formulation of (2.1) is stated as follows: For each $t > 0$, find $u(t) \in H_0^1(\Omega)$ such that

$$\begin{aligned} (u_t(t),\varphi) + \mathcal{A}(u(t),\varphi) &= (f(t),\varphi) \quad \forall \varphi \in H_0^1(\Omega),\ t > 0, \\ u(0) &= u_0(x). \end{aligned}$$

Additionally, we also recall the piecewise linear, continuous finite element space S corresponding to the triangulation $\mathcal{T}_{H,h}$ (as described in Chapter 1) as

$$S := \{v \in C^0(\Omega_{H,h}) \mid v|_K \in \mathbb{P}_1 \ \ \forall K \in \mathcal{T}_{H,h}\},$$

where $\Omega_{H,h} := \operatorname{int}\left(\bigcup_{K\in\mathcal{T}_{H,h}} K\right)$. The CFE space S^{CFE} corresponding to the triangulation $\mathcal{T}_{H,h}$ is defined by

$$S^{\mathrm{CFE}} := \{v \in S \mid \exists\, \mathbf{\Phi} \in \mathbb{R}^{\vartheta_{\mathrm{dof}}} \ \text{ with } v(x) = (\mathcal{E}\mathbf{\Phi})_x \ \ \forall x \in \vartheta_{H,h}\},$$

where $\mathbf{\Phi} : \vartheta_{\mathrm{dof}} \to \mathbb{R}$ is the grid function and the extrapolation operator $\mathcal{E} : \mathbb{R}^{\vartheta_{\mathrm{dof}}} \to \mathbb{R}^{\vartheta_{H,h}}$ is defined in (1.11). The spatially semidiscrete CFE approximation $u^{\mathrm{CFE}}(t) := u^{\mathrm{CFE}}(\cdot,t)$ is to find $u^{\mathrm{CFE}}(t) \in S^{\mathrm{CFE}}$ such that

$$\begin{aligned} (u_t^{\mathrm{CFE}},\chi) + \mathcal{A}(u^{\mathrm{CFE}},\chi) &= (f,\chi) \quad \forall \chi \in S^{\mathrm{CFE}},\ t > 0, \\ u^{\mathrm{CFE}}(0) &= u_0^{\mathrm{CFE}}, \end{aligned} \tag{2.2}$$

where u_0^{CFE} is some suitable approximation of u_0 in S^{CFE}.

For purely parabolic problem, *a priori* error analysis for the spatially semidiscrete and the fully discrete finite element approximations in convex domains has been studied by Thomée in [102]. In order to estimate the errors, the author has used the elliptic or Ritz projection as an intermediate solution and then adopted an error splitting technique in the error analysis to derive *a priori* error bounds in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms. For the smooth initial data, optimal order error estimates in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms are shown to hold for both continuous time spatially semidiscrete and fully discrete FEM, where fully discrete FEM is based on the backward Euler and Crank-Nicolson approximations. In [86], Rech *et al.* have investigated the two-scale CFE method for the Poisson equation in two-dimensional convex domains with homogeneous Dirichlet boundary condition. They have established the approximation and convergence properties of the CFEs in the framework of an *a priori* analysis. The authors of [86] have studied *a priori* error analysis between the solutions of the continuous problem and the semidiscrete problem in the L^2 and H^1-norms and proved optimal order estimates (up to logarithmic terms).

In this chapter, an attempt has been made to extend two-scale CFE method for elliptic problems to parabolic problems in a convex polygonal domain with smooth initial data. The key technical tool for the error estimates include the estimates for the associated elliptic or Ritz projection in the framework of CFE method.

The rest of this chapter is organized as follows. In Section 2.2, we introduce some geometric constants which will be used in the error analysis. The existence and uniqueness of CFE solutions is discussed in this section. Optimal order error estimates (up to logarithmic terms) for the spatially semidiscrete and fully discrete CFE methods for problem (2.1) with smooth initial data are carried out in Section 2.4. Finally, we give some concluding remarks in Section 2.5.

2.2 Some Geometric Constants

Following Rech *et al.* [86], we now define some geometric constants which will be used in the error analysis throughout the thesis.

- Let $K \in \mathcal{T}_\Gamma$. For any pair of vertices $x, y \in \mathbf{G}(\tau)$, where $\tau \in \text{sons}(K)$, let $R_{\tau,x,y}$ denote the minimal rectangle containing x^Γ, y^Γ and τ with one side being parallel to $\overline{x^\Gamma y^\Gamma}$. The alignment condition is skipped when $x^\Gamma = y^\Gamma$. Let the symbol R_τ denote the minimal rectangle which contains $\bigcup_{x,y \in \mathbf{G}(\tau)} R_{\tau,x,y}$. Define the constant C_R by

  $$C_R := \max_{K \in \mathcal{T}_\Gamma} \max_{\tau \in \text{sons}(K)} \left(\frac{\text{diam}\, R_\tau}{h_\tau} \right), \tag{2.3}$$

  where the condition (1.9) implies that $C_R = \mathcal{O}(1)$.

- For $K \in \mathcal{T}_\Gamma$, let B_K denote the minimal ball which contains the set

  $$K \cup \left(\bigcup_{x \in \vartheta_{\text{slave}} \cap K} (T_{x,\Delta_x} \cup T_{x^\Gamma,\Delta_x}) \right) \cup \left(\bigcup_{\tau \in \text{sons}(K)} R_\tau \right),$$

  where for a triangle K and a point $x \in \mathbb{R}^2$, we denote $T_{x,K}$ by the triangle T_S. Here the index S is considered as the minimal number such that T_S contains x and K. We set $B_K = K$ for $K \in \mathcal{T}_H^{\text{in}}$. The constant σ_{uni}, which describes the local quasi-uniformity of the initial overlapping mesh $\mathcal{T}_H$ near the boundary, is defined by

  $$\sigma_{\text{uni}} := \max_{K \in \mathcal{T}_\Gamma} \max_{\substack{\tau \in \mathcal{T}_H \\ \tau \cap B_K \neq \emptyset}} \left(\frac{\text{diam}\, B_K}{h_\tau} \right). \tag{2.4}$$

- For the near-boundary triangles $K \in \mathcal{T}_\Gamma$, we decompose the approximation error estimates into a sum of error estimates on the sons, $\tau \in \text{sons}(K)$. For each $\tau \in \text{sons}(K)$, these estimates involve the given function in the neighborhood of R_τ of τ. For this, a quantity will enter in the error estimates which measures the overlap of such neighborhoods. Keeping this in mind we define the set $\mathcal{T}_{\text{ol}}(\tau)$ for $K \in \mathcal{T}_\Gamma$ and $\tau \in \text{sons}(K)$, as

$$\mathcal{T}_{\text{ol}}(\tau) := \{\tilde{\tau} \in \text{sons}(K) \mid R_{\tilde{\tau}} \cap \tau \neq \emptyset\}.$$

The number of elements in $\mathcal{T}_{\text{ol}}(\tau)$ is finite and this is estimated by

$$\sharp\, \mathcal{T}_{\text{ol}}(\tau) \leq C(1 + \log(h_K/h_\tau)), \tag{2.5}$$

where the constant C depends on the constant C_R defined in (2.3) and on the shape regularity of the mesh. For a proof of (2.5), see [86, Lemma 2].

- To measure $\sharp\, \mathcal{T}_{\text{ol}}(\tau)$ (the cardinality of the set $\mathcal{T}_{\text{ol}}(\tau)$) globally, we introduce the minimal constant $C_{\text{ol},1}$ such that

$$\sharp\, \mathcal{T}_{\text{ol}}(\tau) \leq C_{\text{ol},1} \max_{\tau \in \text{sons}(K)} (1 + \log(h_K/h_\tau)) := \widetilde{\log}(h_K/h_K^{\min}) \quad \forall K \in \mathcal{T}_\Gamma$$

holds, where $h_K^{\min} := \min\limits_{\tau \in \text{sons}(K)} h_\tau$. For $K \in \mathcal{T}_H^{\text{in}}$, we take $\widetilde{\log}(h_K/h_K^{\min}) := 1$. It's global analogue is given by

$$\widetilde{\text{Log}}(H/h) := \max\{\widetilde{\log}(h_K/h_K^{\min}) : K \in \mathcal{T}_\Gamma\}.$$

- The second overlap constant $C_{\text{ol},2}$ is defined by

$$C_{\text{ol},2} := \max_{\tau \in \mathcal{T}_\Gamma} \sharp\, \{K \in \mathcal{T}_H : |B_K \cap \tau| > 0\},$$

where the constant $C_{\text{ol},2}$ is related to the constant σ_{uni} as in (2.4).

- The error estimates for CFEs rely on the existence of an appropriate extension operator on the domain Ω. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, there exists a continuous, linear extension operator $\mathscr{E} : H^k(\Omega) \to H^k(\mathbb{R}^2)$, $k \in \mathbb{N}$, such that

$$\forall u \in H^k(\Omega) : \quad \mathscr{E}u|_\Omega \equiv u \quad \text{and} \quad \|\mathscr{E}u\|_{H^k(\mathbb{R}^2)} \leq \sigma_{\text{ext}} \|u\|_{H^k(\Omega)}, \tag{2.6}$$

where the constant σ_{ext} depending only on k and Ω (cf. [97]). All details including the characterization of the class of domain geometries can be found in [90, 91].

2.3 Existence and Uniqueness of CFE Solution

In order to verify the existence and uniqueness of the solution of (2.2), we need to construct basis functions for the CFE space. For this purpose, we define the continuous and piecewise linear finite element space with respect to the mesh $\mathcal{T}_H^{\text{in}}$ as

$$S^{\text{in}} := \{u \in C^0(\Omega^{\text{in}}) \mid u|_K \in \mathbb{P}_1 \;\; \forall K \in \mathcal{T}_H^{\text{in}}\},$$

where the interior $\Omega^{\text{in}} := \text{int}\left(\bigcup_{K\in\mathcal{T}_H^{\text{in}}} K\right)$. The extrapolation operator defined in (1.11) can be written as an operator $\mathcal{E} : S^{\text{in}} \to S$ in terms of the corresponding spaces such that

$$S^{\text{CFE}} = \mathcal{E}(S^{\text{in}}) \subset S.$$

Let $\{\varrho_i\}_{i=1}^{N^{\text{CFE}}}$ be the standard nodal basis function of the finite element space S^{in}, where $N^{\text{CFE}} = \dim(S^{\text{CFE}})$ with the property

$$\varrho_i(x_j) = \delta_{ij} \quad \forall\, 1 \le i, j \le N^{\text{CFE}}, \tag{2.7}$$

where $\vartheta_{\text{dof}} = \{x_j \,:\, 1 \le j \le N^{\text{CFE}}\}$ are the free nodes in $\mathcal{T}_H^{\text{in}}$ and

$$\delta_{ij} = \begin{cases} 1 & \text{if } \; i = j, \\ 0 & \text{if } \; i \neq j. \end{cases}$$

Let $\{\phi_i^{\text{CFE}}\}_{i=1}^{N^{\text{CFE}}}$ be the nodal basis functions for S^{CFE}, is defined by

$$\phi_i^{\text{CFE}} := \mathcal{E}[\varrho_i] \in S^{\text{CFE}} \quad \forall\, 1 \le i \le N^{\text{CFE}}.$$

Now, with respect to each free node $x_i \in \vartheta_{\text{dof}}$, we can associate a corresponding basis function ϕ_i^{CFE} as in the standard FEM. Therefore, we have

$$\phi_i^{\text{CFE}}(x_j) = \delta_{ij} \quad \forall\, 1 \le i, j \le N^{\text{CFE}}.$$

For each t, $u^{\text{CFE}}(t) = u^{\text{CFE}}(\cdot, t) \in S^{\text{CFE}}$ can be represent as

$$u^{\text{CFE}}(x,t) := \sum_{j=1}^{N^{\text{CFE}}} \alpha_j(t)\phi_j^{\text{CFE}}(x).$$

Therefore, the semidiscrete CFE approximation may be stated as follows: Find the coefficients $\alpha_j(t)$ such that

$$\sum_{j=1}^{N^{\text{CFE}}} \alpha_j'(t)(\phi_j^{\text{CFE}}, \phi_k^{\text{CFE}}) + \sum_{j=1}^{N^{\text{CFE}}} \alpha_j(t)(\nabla\phi_j^{\text{CFE}}, \nabla\phi_k^{\text{CFE}})$$
$$= (f, \phi_k^{\text{CFE}}), \quad k = 1, 2, ..., N^{\text{CFE}},$$

with $\alpha_j(0) = \gamma_j$ for $j = 1, 2, ..., N^{\text{CFE}}$, where γ_j's are the components of the given initial approximation of u_0^{CFE} and $N^{\text{CFE}} = \dim(S^{\text{CFE}})$. In matrix notation, we can express this as

$$A^{\text{CFE}}\,\alpha'(t) + B^{\text{CFE}}\,\alpha(t) = f^{\text{CFE}}(t) \quad \text{for } t > 0, \text{ with } \alpha(0) = \gamma,$$

where $A^{\text{CFE}} = (A_{jk}^{\text{CFE}})$ is the *mass* matrix with elements

$$A_{jk}^{\text{CFE}} = (\phi_j^{\text{CFE}}, \phi_k^{\text{CFE}}),$$

$B^{\text{CFE}} = (B_{jk}^{\text{CFE}})$ is the *stiffness* matrix with entries

$$B_{jk}^{\text{CFE}} = (\nabla\phi_j^{\text{CFE}}, \nabla\phi_k^{\text{CFE}}),$$

$f^{\text{CFE}} = (f_k^{\text{CFE}})$ is the vector with entries $f_k^{\text{CFE}} = (f, \phi_k^{\text{CFE}})$ and $\gamma = (\gamma_k)$. Since A^{CFE} is positive definite and hence invertible. Therefore, the resulting system of ordinary differential equations

$$\alpha'(t) + (A^{\text{CFE}})^{-1}B^{\text{CFE}}\,\alpha(t) = (A^{\text{CFE}})^{-1}f^{\text{CFE}}(t) \quad \text{with } \alpha(0) = \gamma,$$

has a unique solution for $t > 0$.

2.4 CFE Error Estimates

This section is concerned with the semidiscrete CFE error analysis for smooth initial data. The convergence properties for the CFE solutions in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms are established.

For the purpose of error analysis of the *semidiscrete* problem, we introduce so-called *elliptic* or *Ritz projection* R^{CFE} onto S^{CFE} ($R^{\text{CFE}} : H_0^1(\Omega) \to S^{\text{CFE}}$) as the orthogonal projection with respect to the inner product $(\nabla v, \nabla w)$ so that

$$(\nabla R^{\text{CFE}}v, \nabla\chi) = (\nabla v, \nabla\chi) \quad \forall\chi \in S^{\text{CFE}}, \quad \text{for } v \in H_0^1(\Omega). \tag{2.8}$$

It is easy to verify the stability of the elliptic projection in H^1-norm, i.e.,

$$\|\nabla R^{\text{CFE}}v\| \le \|\nabla v\| \quad \forall v \in H_0^1(\Omega).$$

2.4.1 Spatially Semidiscrete Error Estimates

For the purpose of spatially semidiscrete error analysis we split the error $u^{\text{CFE}}(t) - u(t)$ as

$$u^{\text{CFE}}(t) - u(t) = \theta(t) + \rho(t), \quad \text{where } \theta(t) = u^{\text{CFE}}(t) - R^{\text{CFE}}u(t),$$
$$\rho(t) = R^{\text{CFE}}u(t) - u(t).$$

In the following lemma we estimate the errors in the Ritz projection in the framework of CFE method.

Lemma 2.4.1. *Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and R^{CFE} be defined by (2.8). Suppose that the conditions (1.9) and*

$$dist(x, \Delta_x) \le \sigma_e h_{\Delta_x} \quad \forall x \in \vartheta_{slave} \tag{2.9}$$

hold true. Then there exists a positive constant C depends on σ_{dist}, σ_e, σ_{uni}, $C_{ol,1}$, $C_{ol,2}$, σ_{ext} and the minimal angles in the triangulation $\mathcal{T}_{H,h}$, such that

$$\|\nabla(R^{CFE}u - u)\| \le CH\widetilde{Log}^{1/2}(H/h)\|u\|_{2,\Omega}, \tag{2.10}$$

and

$$\|R^{CFE}u - u\| \le CH^2\widetilde{Log}(H/h)\|u\|_{2,\Omega}. \tag{2.11}$$

Proof. We begin with the proof of (2.10). For $u \in H_0^1(\Omega) \cap H^2(\Omega)$, define the grid function $\bar{\mathbf{u}} : \vartheta_{\text{dof}} \to \mathbb{R}$ by $\bar{\mathbf{u}}_x := u(x)$, $x \in \vartheta_{\text{dof}}$. Let $R^{\text{CFE}}u$ be the $\mathbb{P}_1-$ nodal interpolant of $\mathcal{E}\bar{\mathbf{u}}$ on $\mathcal{T}_{H,h}$, where the extrapolation operator $\mathcal{E}$ is defined in (1.11). We identify u with its extension $\mathscr{E}u$. Based on the two-scale grid discretization, two cases can arise:

Case I: For any $K \in \mathcal{T}_H^{\text{in}}$, the function $R^{\text{CFE}}u|_K$ is the restriction of $R^{\text{CFE}}u$ on K of the values $(u(x))_{x\in\mathbf{G}(K)}$. Hence, the estimate (2.10) follows from the standard interpolation estimate (see e.g., [25]).

Case II: Now consider $K \in \mathcal{T}_\Gamma$. In order to estimate (2.10), we recall (1.8) which defines the set of sons. For any $\tau \in \text{sons}(K)$, we can write

$$\begin{aligned}\|R^{\text{CFE}}u - u\|_{1,\Omega}^2 &\le \sum_{K\in\mathcal{T}_H}\sum_{\tau\in\text{sons}(K)} \|R^{\text{CFE}}u - u\|_{1,\tau}^2 \\ &\le \sum_{K\in\mathcal{T}_H}\sum_{\tau\in\text{sons}(K)} \Big[\|R^{\text{CFE}}u - \mathcal{I}_\tau u\|_{1,\tau}^2 + \|\mathcal{I}_\tau u - u\|_{1,\tau}^2\Big],\end{aligned} \tag{2.12}$$

where $\mathcal{I}_\tau : C^0(\tau) \to \mathbb{P}_1(\tau)$ is the Lagrange linear interpolation operator on τ. To estimate the two terms of the right-hand side of (2.12), we argue as in [86, Theorem 1] and use (2.6) to obtain

$$\begin{aligned}\|R^{\text{CFE}}u - u\|_{1,\Omega}^2 &\le C\widetilde{\text{Log}}(H/h)\sum_{K\in\mathcal{T}_H} h_K^2\|\mathcal{E}u\|_{2,B_K}^2 \\ &\le CC_{\text{ol},2}\widetilde{\text{Log}}(H/h)H^2\sum_{\tau\in\mathcal{T}_H}\|\mathcal{E}u\|_{2,\tau}^2 \\ &\le CC_{\text{ol},2}\sigma_{\text{ext}}^2\widetilde{\text{Log}}(H/h)H^2\|u\|_{2,\Omega}^2,\end{aligned}$$

which shows (2.10). To prove (2.11), we use the duality argument. For $\varphi \in L^2(\Omega)$, let $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of

$$-\Delta\psi = \varphi \quad \text{in } \Omega, \qquad \text{with} \quad \psi = 0 \quad \text{on } \Gamma.$$

Setting $\chi = R^{\mathrm{CFE}}\psi$, we obtain

$$\begin{aligned}
(R^{\mathrm{CFE}}u - u, \varphi) &= (\nabla(R^{\mathrm{CFE}}u - u), \nabla\psi) \\
&= (\nabla(R^{\mathrm{CFE}}u - u), \nabla(\psi - \chi)) \\
&\leq \|\nabla(R^{\mathrm{CFE}}u - u)\|\|\nabla(\psi - \chi)\| \\
&\leq (CH\widetilde{\mathrm{Log}}^{1/2}(H/h)\|u\|_2)(CH\widetilde{\mathrm{Log}}^{1/2}(H/h)\|\psi\|_2) \\
&\leq CH^2\widetilde{\mathrm{Log}}(H/h)\|u\|_2\|\varphi\|,
\end{aligned}$$

which gives

$$\|R^{\mathrm{CFE}}u - u\| \leq CH^2\widetilde{\mathrm{Log}}(H/h)\|u\|_{2,\Omega},$$

and this completes the proof of the lemma. □

Remark 2.4.1. *We rewrite the equation (2.8) for $v = u$ as*

$$(\nabla R^{CFE}u, \nabla\chi) = (\nabla u, \nabla\chi) \quad \textit{for } u \in H_0^1(\Omega),$$

differentiating both side with respect to 't', we have

$$(\nabla R^{CFE}u_t, \nabla\chi) = (\nabla u_t, \nabla\chi).$$

Analogous to the proof of Lemma 2.4.1 and for $u_t \in H_0^1(\Omega) \cap H^2(\Omega)$, it is easy to obtain a bound for the temporal derivative of $(R^{CFE}u - u)$ as

$$\|\nabla(R^{CFE}u_t - u_t)\| \leq CH\widetilde{Log}^{1/2}(H/h)\|u_t\|_{2,\Omega}, \tag{2.13}$$

and

$$\|R^{CFE}u_t - u_t\| \leq CH^2\widetilde{Log}(H/h)\|u_t\|_{2,\Omega}. \tag{2.14}$$

The main result for the semidiscrete CFE error estimate is given in the following theorem.

Theorem 2.4.1. *Let $u^{CFE}(t)$ and $u(t)$ be the solutions of (2.2) and (2.1), respectively, and $u(t)$, $u_t(t) \in H_0^1(\Omega) \cap H^2(\Omega)$ for each t. Also, let the conditions (1.9) and (2.9)* *hold. Then there exists a positive constant C independent of (h, H) such that, for $t \geq 0$, we have*

$$\|u^{CFE}(t) - u(t)\| \leq \|u_0^{CFE} - u_0\| + CH^2\widetilde{Log}(H/h)\left(\|u_0\|_{2,\Omega} + \|u(t)\|_{2,\Omega} + \int_0^t \|u_t\|_{2,\Omega} d\zeta\right), \tag{2.15}$$

and

$$\|\nabla(u^{CFE}(t) - u(t))\| \leq C\|\nabla(u_0^{CFE} - u_0)\| + CH\widetilde{Log}^{1/2}(H/h)\left(\|u_0\|_{2,\Omega} + \|u(t)\|_{2,\Omega} + (\int_0^t \|u_t\|^2_{2,\Omega} d\zeta)^{1/2}\right).$$

Proof. We closely follow the proof technique of [102]. Since $\rho(t)$ is bounded by Lemma 2.4.1, it only remains to bound $\theta(t)$. It is quite standard to verify that $\theta(t)$ satisfies the error equation of the form

$$(\theta_t, \chi) + \mathcal{A}(\theta, \chi) = -(\rho_t, \chi) \quad \forall \chi \in S^{\text{CFE}},\ t > 0, \tag{2.16}$$

where we have used the fact that the operator R^{CFE} commutes with respect to the time differentiation. Since $\theta \in S^{\text{CFE}}$, we choose $\chi = \theta$ in (2.16) and obtain

$$(\theta_t, \theta) + \|\nabla\theta\|^2 = -(\rho_t, \theta) \quad \text{for } t > 0. \tag{2.17}$$

The first term $(\theta_t, \theta) = \frac{1}{2}\frac{d}{dt}\|\theta\|^2$ and the second term is nonnegative. Since $\|\theta\|$ might not be differentiable when $t = 0$, we obtain from (2.17)

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = \frac{1}{2}\frac{d}{dt}(\|\theta\|^2 + \epsilon^2) \leq \|\rho_t\|\|\theta\|,$$

with $\epsilon > 0$ and hence

$$(\|\theta\|^2 + \epsilon^2)^{1/2}\frac{d}{dt}(\|\theta\|^2 + \epsilon^2)^{1/2} \leq \|\rho_t\|\|\theta\|.$$

Using $\|\theta\| \leq (\|\theta\|^2 + \epsilon^2)^{1/2}$, we obtain

$$\frac{d}{dt}(\|\theta\|^2 + \epsilon^2)^{1/2} \leq \|\rho_t\|.$$

Integrating from 0 to t and letting $\epsilon \to 0$, it now follows that

$$\|\theta(t)\| \leq \|\theta(0)\| + \int_0^t \|\rho_t\| d\zeta. \tag{2.18}$$

Since

$$\|\theta(0)\| = \|u_0^{\mathrm{CFE}} - R^{\mathrm{CFE}}u_0\| \leq \|u_0^{\mathrm{CFE}} - u_0\| + \|R^{\mathrm{CFE}}u_0 - u_0\| = \|u_0^{\mathrm{CFE}} - u_0\| + \|\rho(0)\|,$$

we obtain from (2.18)

$$\|\theta(t)\| \leq \|u_0^{\mathrm{CFE}} - u_0\| + \|\rho(0)\| + \int_0^t \|\rho_t\| d\zeta. \tag{2.19}$$

Note that $\rho(0)$ and ρ_t are bounded as desired by Lemma 2.4.1 and (2.14), respectively. Using the bounds for $\rho(0)$ and ρ_t together with (2.19) we obtain the required estimate (2.15).

Next, to prove the gradient error estimates we split the error term as before

$$\nabla(u^{\mathrm{CFE}}(t) - u(t)) = \nabla\theta(t) + \nabla\rho(t),$$

and $\nabla\rho(t)$ is bounded by Lemma 2.4.1. To estimate $\nabla\theta(t)$ we choose this time $\chi = \theta_t$ in (2.16) to obtain

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 = -(\rho_t, \theta_t) \leq \frac{1}{2}\|\rho_t\|^2 + \frac{1}{2}\|\theta_t\|^2,$$

so that $\frac{d}{dt}\|\nabla\theta\|^2 \leq \|\rho_t\|^2$, and simplifying

$$\begin{aligned}\|\nabla\theta(t)\|^2 &\leq \|\nabla\theta(0)\|^2 + \int_0^t \|\rho_t\|^2 d\zeta \\ &\leq (\|\nabla(u_0^{\mathrm{CFE}} - u_0)\| + \|\nabla(R^{\mathrm{CFE}}u_0 - u_0)\|)^2 + \int_0^t \|\rho_t\|^2 d\zeta,\end{aligned}$$

which gives, using (2.14) and Lemma 2.4.1,

$$\|\nabla\theta(t)\|^2 \leq 2\|\nabla(u_0^{\mathrm{CFE}} - u_0)\|^2 + CH^2\widetilde{\mathrm{Log}}(H/h)\left(\|u_0\|_{2,\Omega}^2 + \int_0^t \|u_t\|_{2,\Omega}^2 d\zeta\right),$$

and this completes the proof. $\square$

2.4.2 Fully Discrete Error Estimates

In this section, we study some simple schemes for discretization with respect to time variable. We derive error estimates for the application of the *backward Euler* and the *Crank-Nicolson* methods to the discretization with respect to time variable of the spatially semidiscrete problem (2.2).

Backward Euler method with constant time step: Let k be the constant time step and let U^n be the approximation of the exact solution $u(t)$ of (2.1) at $t = t_n = nk$ in

S^{CFE}, where n is a nonnegative integer. Setting $\bar{\partial}U^n = (U^n - U^{n-1})/k$, we define the backward Euler method by replacing the time derivative in (2.2) as

$$(\bar{\partial}U^n, \chi) + \mathcal{A}(U^n, \chi) = (f(t_n), \chi) \quad \forall \chi \in S^{\text{CFE}},\ n \geq 1, \tag{2.20}$$
$$\text{with} \quad U^0 = u_0^{\text{CFE}}.$$

The above equation can be written as

$$(U^n, \chi) + k\mathcal{A}(U^n, \chi) = (U^{n-1} + kf(t_n), \chi) \quad \forall \chi \in S^{\text{CFE}}.$$

Now, with notations as in the semidiscrete case, we express the above equation in vector and matrix notations as

$$(A^{\text{CFE}} + kB^{\text{CFE}})\alpha^n = A^{\text{CFE}}\alpha^{n-1} + kf^{\text{CFE}}(t_n),$$

where $A^{\text{CFE}} + kB^{\text{CFE}}$ is positive definite and hence invertible.

We shall prove the following error estimate.

Theorem 2.4.2. *Let U^n and u be the solutions of (2.20) and (2.1), respectively with $u, u_t \in H_0^1(\Omega) \cap H^2(\Omega)$. Then*

$$\|U^n - u(t_n)\| \leq \|u_0^{CFE} - u_0\| + CH^2\widetilde{Log}(H/h)\left(\|u_0\|_{2,\Omega} + \int_0^{t_n} \|u_t\|_{2,\Omega} d\zeta\right) + k\int_0^{t_n} \|u_{tt}\| d\zeta \quad \text{for } n \geq 0.$$

Proof. We first write

$$U^n - u(t_n) = (U^n - R^{\text{CFE}}u(t_n)) + (R^{\text{CFE}}u(t_n) - u(t_n)) = \theta^n + \rho^n.$$

Since ρ^n is bounded in view of Lemma 2.4.1, we only need to estimate θ^n. It is easy to verify that θ^n satisfies the following equation

$$(\bar{\partial}\theta^n, \chi) + \mathcal{A}(\theta^n, \chi) = -(\omega^n, \chi) \quad \forall \chi \in S^{\text{CFE}},\ n \geq 1, \tag{2.21}$$

where

$$\omega^n = R^{\text{CFE}}\bar{\partial}u(t_n) - u_t(t_n) = (R^{\text{CFE}} - I)\bar{\partial}u(t_n) + (\bar{\partial}u(t_n) - u_t(t_n)) = \omega_1^n + \omega_2^n.$$

With $\chi = \theta^n$ in (2.21) we obtain

$$\|\theta^n\| \leq \|\theta^{n-1}\| + k\|\omega^n\|,$$

a repeated application yields

$$\|\theta^n\| \le \|\theta^0\| + k\sum_{j=1}^{n} \|\omega_1^j\| + k\sum_{j=1}^{n} \|\omega_2^j\|. \tag{2.22}$$

Now, we need to estimate all the three terms appearing on the right-hand side separately. The first term is estimated as

$$\begin{aligned} \|\theta^0\| = \|\theta(0)\| &\le \|u_0^{\text{CFE}} - u_0\| + \|R^{\text{CFE}}u_0 - u_0\| \\ &\le \|u_0^{\text{CFE}} - u_0\| + CH^2\widetilde{\text{Log}}(H/h)\|u_0\|_{2,\Omega}. \end{aligned}$$

Following the arguments of [102], the second and third terms in (2.22) can be bounded as

$$k\sum_{j=1}^{n}\|\omega_1^j\| \le \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} CH^2\widetilde{\text{Log}}(H/h)\|u_t\|_{2,\Omega}d\zeta = CH^2\widetilde{\text{Log}}(H/h)\int_0^{t_n}\|u_t\|_{2,\Omega}d\zeta.$$

$$k\sum_{j=1}^{n}\|\omega_2^j\| \le \sum_{j=1}^{n}\left\|\int_{t_{j-1}}^{t_j}(\zeta - t_{j-1})u_{tt}(\zeta)d\zeta\right\| \le k\int_0^{t_n}\|u_{tt}\|d\zeta.$$

Altogether these estimates complete the proof of the theorem. □

Remark 2.4.2. *For the gradient norm error estimate we write $\nabla(U^n - u(t_n)) = \nabla\theta^n + \nabla\rho^n$, where $\nabla\rho^n$ is bounded as claimed in Lemma 2.4.1.*

To estimate $\nabla\theta$, choose $\chi = \bar{\partial}\theta^n$ in (2.21) to obtain

$$\bar{\partial}\|\nabla\theta^n\|^2 \le \|\omega^n\|^2,$$

and if $\nabla\theta^0 = 0$, this gives

$$\|\nabla\theta^n\|^2 \le k\sum_{j=1}^{n}\|\omega^j\|^2 \le Ck\sum_{j=1}^{n}\|\omega_1^j\|^2 + Ck\sum_{j=1}^{n}\|\omega_2^j\|^2,$$

using the estimates for ω_1^j and ω_2^j as before and together with the estimates $\nabla\rho^n$ we have

$$\|\nabla(U^n - u(t_n))\| \le C(u)(H\widetilde{Log}^{1/2}(H/h) + k).$$

So far we have used a constant time step k in the above time discretization scheme. We now introduce the fully discrete method with a variable time step of the backward Euler method.

Backward Euler method with variable time step: Let $0 = t_0 < t_1 < \cdots < t_n < \cdots$ be a partition of the positive time axis and let k_n be the variable time step defined by

$k_n = t_n - t_{n-1}$. Let U^n be the approximation of $u(t_n)$. Thus (2.20) will be replaced by

$$(\bar{\partial}_n U^n, \chi) + \mathcal{A}(U^n, \chi) = (f(t_n), \chi) \quad \forall \chi \in S^{\mathrm{CFE}},\ n \geq 1, \tag{2.23}$$
$$\text{with} \quad U^0 = u_0^{\mathrm{CFE}},$$

where $\bar{\partial}_n U^n = (U^n - U^{n-1})/k_n$. We shall prove the following error estimate which reduces to that of Theorem 2.4.2 for constant time steps with $k_n = k$.

Theorem 2.4.3. *Let U^n and u be the solutions of (2.23) and (2.1), respectively with $u, u_t \in H_0^1(\Omega) \cap H^2(\Omega)$. Then, we have for $n \geq 0$,*

$$\|U^n - u(t_n)\| \leq \|u_0^{CFE} - u_0\| + CH^2 \widetilde{Log}(H/h)\Big(\|u_0\|_{2,\Omega} + \int_0^{t_n} \|u_t\|_{2,\Omega} d\zeta\Big) + \sum_{j=1}^{n} k_j \int_{t_{j-1}}^{t_j} \|u_{tt}\| d\zeta.$$

Proof. We first write $U^n - u^n = \theta^n + \rho^n$. This time θ^n satisfies

$$(\bar{\partial}_n \theta^n, \chi) + \mathcal{A}(\theta^n, \chi) = -(\omega^n, \chi) \quad \forall \chi \in S^{\mathrm{CFE}},\ n \geq 1,$$

where

$$\omega^n = (R^{\mathrm{CFE}} - I)\bar{\partial}_n u^n + (\bar{\partial}_n u^n - u_t^n) = \omega_1^n + \omega_2^n.$$

Analogous to (2.22), we obtain

$$\|\theta^n\| \leq \|\theta^0\| + \sum_{j=1}^{n} k_j \big(\|\omega_1^j\| + \|\omega_2^j\|\big).$$

As before, we estimate

$$\sum_{j=1}^{n} k_j \|\omega_1^j\| \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} CH^2 \widetilde{\mathrm{Log}}(H/h) \|u_t\|_{2,\Omega} d\zeta \leq CH^2 \widetilde{\mathrm{Log}}(H/h) \int_0^{t_n} \|u_t\|_{2,\Omega} d\zeta,$$

and

$$\sum_{j=1}^{n} k_j \|\omega_2^j\| \leq \sum_{j=1}^{n} \Big\| \int_{t_{j-1}}^{t_j} (\zeta - t_{j-1}) u_{tt}(\zeta) d\zeta \Big\| \leq \sum_{j=1}^{n} k_j \int_{t_{j-1}}^{t_j} \|u_{tt}\| d\zeta.$$

Now, the above estimates together with the standard estimates for ρ^n and θ^0 completes the proof. □

Next, we turn to the Crank-Nicolson CFE method which yields the second order accuracy in k.

Crank-Nicolson method: In the Crank-Nicolson method, the semidiscrete equation (2.2) is discretized in a symmetric way around the point $t_{n-\frac{1}{2}} = (n-\frac{1}{2})k$. In this case, for U^n in S^{CFE}, $n \geq 1$, we have

$$(\bar{\partial}U^n, \chi) + \mathcal{A}(\frac{U^n + U^{n-1}}{2}, \chi) = (f(t_{n-\frac{1}{2}}), \chi) \quad \forall \chi \in S^{\text{CFE}},\ n \geq 1, \tag{2.24}$$

$$\text{with} \quad U^0 = u_0^{\text{CFE}},$$

which can be put in matrix and vector form as

$$(A^{\text{CFE}} + \frac{1}{2}kB^{\text{CFE}})\alpha^n = (A^{\text{CFE}} - \frac{1}{2}kB^{\text{CFE}})\alpha^{n-1} + kf^{\text{CFE}}(t_{n-\frac{1}{2}}).$$

Since $A^{\text{CFE}} + \frac{1}{2}kB^{\text{CFE}}$ is positive definite and hence, in particular, invertible. This proves the existence and uniqueness of U^n.

We now have the following theorem.

Theorem 2.4.4. *Let U^n and u be the solutions of (2.24) and (2.1), respectively with $u, u_t \in H_0^1(\Omega) \cap H^2(\Omega)$. Then, we have for $n \geq 0$*

$$\|U^n - u(t_n)\| \leq \|u_0^{CFE} - u_0\| + CH^2\widetilde{Log}(H/h)\left(\|u_0\|_{2,\Omega} + \int_0^{t_n} \|u_t\|_{2,\Omega} d\zeta\right) + Ck^2\int_0^{t_n}(\|u_{ttt}\| + \|\Delta u_{tt}\|)d\zeta.$$

Proof. Since ρ^n is bounded as above, we will estimate only θ^n. In this case, we have

$$(\bar{\partial}\theta^n, \chi) + \mathcal{A}(\frac{\theta^n + \theta^{n-1}}{2}, \chi) = -(\omega^n, \chi) \quad \forall \chi \in S^{\text{CFE}},\ n \geq 1, \tag{2.25}$$

where

$$\begin{aligned}\omega^n &= (R^{\text{CFE}} - I)\bar{\partial}u(t_n) + (\bar{\partial}u(t_n) - u_t(t_{n-\frac{1}{2}})) - \Delta\big(u(t_{n-\frac{1}{2}}) - \frac{1}{2}(u(t_n) + u(t_{n-1}))\big)\\ &= \omega_1^n + \omega_2^n + \omega_3^n.\end{aligned}$$

Now, choosing $\chi = \frac{\theta^n + \theta^{n-1}}{2}$ in (2.25), we obtain

$$\|\theta^n\| \leq \|\theta^0\| + k\sum_{j=1}^{n}\big(\|\omega_1^j\| + \|\omega_2^j\| + \|\omega_3^j\|\big),$$

where θ^0 and ω_1^j are estimated as before. To estimate ω_2^j and ω_3^j, we proceed as in [102] to obtain

$$k\sum_{j=1}^{n}\left(\|\omega_2^j\| + \|\omega_3^j\|\right) \leq Ck^2\int_0^{t_n}(\|u_{ttt}\| + \|\Delta u_{tt}\|)d\zeta,$$

and this completes the proof. □

2.5 Concluding Remarks

In this chapter, we proposed a two-scale CFE method for time dependent IBVP (2.1) in a convex domain with smooth initial data. A two-scale CFE discretization is used for the space discretizations, where the coarse-scale grid discretized the domain at an appropriate distance from the boundary and determines the degrees of freedom and the fine-scale grid is used to resolve the boundary. A continuous, piecewise linear CFE space is employed for the spatially semidiscrete finite element approximation and the temporal discretization is based on the backward Euler and the Crank-Nicolson schemes. Optimal order error estimates (up to logarithmic terms) in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms are derived for both spatially semidiscrete and fully discrete CFE methods with the smooth initial data. We observe that when $h = \mathcal{O}(H)$, the two-scale grid $\mathcal{T}_{H,h}$ coincides with the coarse grid $\mathcal{T}_H$ and hence, Theorems 2.4.1, 2.4.2, 2.4.3 and 2.4.4 yields the same ROC as in the case of standard FEM (cf. [102]). Numerical experiment is presented in Chapter 7 (see Example 7.1) to support our theoretical results. In contrast to the standard FEM the $\sharp$dof is much smaller for the CFE method, which shows the efficiency of the CFE method (see the comparison in Tables 7.5 and 7.6 of Chapter 7).

3

Two-Scale CFE Method for Parabolic Problems in Convex Polygonal Domains with Nonsmooth Initial Data

This chapter is concerned with *a priori* error analysis of the spatially semidiscrete and fully discrete CFE approximations of the homogeneous parabolic problem (1.2) in a convex polygonal domain when the given data $u_0 \in L^2(\Omega)$. The eigenfunctions expansion related to the elliptic operator and the rational approximations for exponential are the key technical tools used in nonsmooth data error analysis. The fully discrete error analysis is based on the backward Euler method. *A priori* error estimates in the $L^\infty(L^2)$-norm for positive time is established. Like standard FEM, the analogous behaviour of the CFE solution is studied and an optimal order convergence (up to logarithmic terms) for the solution is derived even for nonsmooth initial data.

3.1 Introduction

We shall begin by recalling the IBVP for a homogeneous parabolic equation of the form

$$
\begin{aligned}
&u_t = \Delta u \quad \text{in } \Omega,\ t > 0, \\
&u = 0 \quad \text{on } \Gamma,\ t > 0, \quad \text{with} \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega,
\end{aligned}
\tag{3.1}
$$

where Ω is a bounded convex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary Γ and the initial data $u_0 \in L^2(\Omega)$. The weak formulation of (3.1) is stated as follows: For each $t > 0$, find $u(t) \in H_0^1(\Omega)$ such that

$$
\begin{aligned}
(u_t(t), \varphi) + \mathcal{A}(u(t), \varphi) &= 0 \quad \forall \varphi \in H_0^1(\Omega),\ t > 0, \\
u(0) &= u_0(x).
\end{aligned}
$$

In order to define the CFE space, let S be the piecewise linear and continuous finite element space corresponding to the triangulation $\mathcal{T}_{H,h}$ as

$$S := \{v \in C^0(\Omega_{H,h}) \ | \ v|_K \in \mathbb{P}_1 \ \ \forall K \in \mathcal{T}_{H,h}\},$$

where $\Omega_{H,h} := \text{int}\left(\bigcup_{K\in\mathcal{T}_{H,h}} K\right)$. Then the CFE space S^{CFE} corresponding to the triangulation $\mathcal{T}_{H,h}$ is given by

$$S^{\text{CFE}} := \{v \in S \mid \exists\, \mathbf{\Phi} \in \mathbb{R}^{\vartheta_{\text{dof}}} \text{ with } v(x) = (\mathcal{E}\mathbf{\Phi})_x \ \ \forall x \in \vartheta_{H,h}\},$$

where $\mathbf{\Phi} : \vartheta_{\text{dof}} \to \mathbb{R}$ is the grid function and the extrapolation operator $\mathcal{E} : \mathbb{R}^{\vartheta_{\text{dof}}} \to \mathbb{R}^{\vartheta_{H,h}}$ is defined in (1.11). Then the spatially semidiscrete CFE approximation $u^{\text{CFE}}(t) := u^{\text{CFE}}(\cdot, t)$ is to find $u^{\text{CFE}}(t) \in S^{\text{CFE}}$ such that

$$\begin{aligned} (u_t^{\text{CFE}}, \chi) + \mathcal{A}(u^{\text{CFE}}, \chi) &= 0 \quad \forall \chi \in S^{\text{CFE}},\ t > 0, \\ u^{\text{CFE}}(0) &= u_0^{\text{CFE}}, \end{aligned} \tag{3.2}$$

where u_0^{CFE} is some suitable approximation of u_0 in S^{CFE}.

Following [102], we introduce certain Hilbert spaces $\dot{H}^\nu(\Omega)$, which are convenient in describing the regularity of the solutions of the given IBVP (3.1). Let $\{\varphi_j\}_{j=1}^\infty$ be the sequence of orthonormal eigenfunctions corresponding to the eigenvalues $\{\lambda_j\}_{j=1}^\infty$ associated with the eigenvalue problem

$$-\Delta\varphi = \lambda\varphi \quad \text{in } \ \Omega, \quad \text{with } \ \varphi = 0 \quad \text{on } \ \Gamma.$$

Since $\{\varphi_j\}_{j=1}^\infty$ form a basis for $L^2(\Omega)$, for each $v \in L^2(\Omega)$, we can write as $v = \sum_{j=1}^\infty (v, \varphi_j)\varphi_j$, and the following relation

$$(v, w) = \sum_{j=1}^\infty (v, \varphi_j)(w, \varphi_j)$$

holds, which is called the Parseval's relation. For $\nu \geq 0$, let $\dot{H}^\nu = \dot{H}^\nu(\Omega)$ be the subspace of $L^2(\Omega)$ defined by

$$\|v\|_{\dot{H}^\nu} = \left(\sum_{j=1}^\infty \lambda_j^\nu (v, \varphi_j)^2\right)^{1/2} < \infty \quad \text{for } \nu \geq 0.$$

The nonnegative powers of the operator $-\Delta$ is given by

$$(-\Delta)^\nu v = \sum_{j=1}^\infty \lambda_j^\nu (v, \varphi_j)\varphi_j \quad \text{for } \nu \geq 0.$$

Thus the norm $\|\cdot\|_{\dot{H}^\nu}$ may be expressed as

$$\|v\|_{\dot{H}^\nu} = \|(-\Delta)^{\nu/2}v\| = ((-\Delta)^\nu v, v)^{1/2}.$$

For ν is a nonnegative integer we have the following characterization: The space

$$\dot{H}^\nu(\Omega) = \{v \in H^\nu(\Omega) : \Delta^j v = 0 \ \text{ on } \ \Gamma, \ \text{ for } \ j < \nu/2\},$$

where the boundary conditions are interpreted in the sense of traces in $L^2(\Gamma)$, and the norms $\|\cdot\|_{\dot{H}^\nu}$ and $\|\cdot\|_{H^\nu}$ are equivalent in $\dot{H}^\nu$ [102, Lemma 3.1].

Let $E(t)$ be the solution operator of problem (3.1). Then the solution may be written as

$$u(t) = E(t)u_0 = \sum_{j=1}^{\infty} e^{-\lambda_j t}(u_0, \varphi_j)\varphi_j. \tag{3.3}$$

Note that, $E(t)$ is an analytic semigroup on $L^2(\Omega)$ generated by the Laplacian Δ (cf. Pazy [74]). For $u_0 \in L^2(\Omega)$ and $t > 0$, the solution $u(t) = E(t)u_0$ of (3.1) belongs to $\dot{H}^\nu(\Omega)$ for $\nu \geq 0$. Further, for $0 \leq \nu \leq q$ and $l \geq 0$, we have

$$\|D_t^l E(t)u_0\|_{\dot{H}^q} \leq Ct^{-(q-\nu)/2-l}\|u_0\|_{\dot{H}^\nu} \quad \text{for} \ \ u_0 \in \dot{H}^\nu(\Omega), \ t > 0. \tag{3.4}$$

For a proof of this inequality, see [102, Lemma 3.2].

Results for nonsmooth data for the homogeneous equation in the framework of FEM were first discussed by spectral representation in Blair [12], Fujita and Mizutani [41] and Thomée [99]. Later, finite element approximation for the homogeneous equation with nonsmooth initial data has been studied by the energy method in Luskin and Rannacher [69], Sammon [88] and Thomée [100]. Thomée [102] has studied *a priori* error analysis for the finite element approximation of the solution of homogeneous parabolic equation in convex domains with nonsmooth initial data and proved that optimal order convergence takes place for positive time even the initial data is nonsmooth.

In this chapter, we derive $L^\infty(L^2)$ *a priori* error bounds for the semidiscrete and fully discrete CFE approximations of homogeneous parabolic equation (3.1) in convex domains with nonsmooth initial data. *A priori* error analysis is based on the eigenfunctions expansion related to the elliptic operator and the rational approximations for exponential.

The layout of this chapter is as follows. Section 3.2.1 is devoted to the spatially semidiscrete error analysis. Section 3.2.2 is concerned with the fully discrete error analysis for the backward Euler method. Finally, some concluding remarks are presented in Section 3.3.

3.2 CFE Error Estimates

In this section, we concentrate on the spatially semidiscrete and fully discrete CFE error analysis for the problem (3.1) with initial data $u_0 \in L^2(\Omega)$. The convergence properties for the CFE solution for positive time in the $L^\infty(L^2)$-norm is established.

3.2.1 Spatially Semidiscrete Error Estimates

For the purpose of spatially semidiscrete error estimate, let T be the exact solution operator of the Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \text{ on } \Gamma,$$

defined by $u = Tf$, and let $\{T^{\text{CFE}}\}$ be a family of solution operators $T^{\text{CFE}} : L^2(\Omega) \to S^{\text{CFE}}$ approximating T such that $u^{\text{CFE}} = T^{\text{CFE}} f \in S^{\text{CFE}}$ and

$$(\nabla u^{\text{CFE}}, \nabla \chi) = (f, \chi) \quad \forall \chi \in S^{\text{CFE}}.$$

Using the definition of T^{CFE}, we have

$$(f, T^{\text{CFE}} g) = (\nabla T^{\text{CFE}} f, \nabla T^{\text{CFE}} g) = (T^{\text{CFE}} f, g) \quad \forall f, g \in L^2(\Omega),$$

and hence,

$$(T^{\text{CFE}} f, f) = \|\nabla T^{\text{CFE}} f\|^2 \geq 0.$$

Thus T^{CFE} is selfadjoint and positive semidefinite on $L^2(\Omega)$. In fact, T^{CFE} is positive definite on S^{CFE}. Further,

$$(\nabla T^{\text{CFE}} f, \nabla \chi) = (f, \chi) = (\nabla T f, \nabla \chi) = (\nabla R^{\text{CFE}} T f, \nabla \chi) \quad \forall \chi \in S^{\text{CFE}},$$

and hence $T^{\text{CFE}} = R^{\text{CFE}} T$. The following properties holds for T^{CFE}:

(a) $\|(T^{\text{CFE}} - T) f\| \leq C H^2 \|f\|$ for $f \in L^2(\Omega)$, and

(b) T^{CFE} is selfadjoint, positive semidefinite on $L^2(\Omega)$ and positive definite on S^{CFE}.

Let $\Delta^{\text{CFE}} : S^{\text{CFE}} \to S^{\text{CFE}}$ be a *discrete Laplacian* defined by

$$(\Delta^{\text{CFE}} w^{\text{CFE}}, \chi) = -(\nabla w^{\text{CFE}}, \nabla \chi) \quad \forall\, w^{\text{CFE}}, \chi \in S^{\text{CFE}}. \tag{3.5}$$

Then, observing the following

$$(f^{\text{CFE}}, \chi) = (\nabla (T^{\text{CFE}} f^{\text{CFE}}), \nabla \chi) = -(\Delta^{\text{CFE}} (T^{\text{CFE}} f^{\text{CFE}}), \chi) \quad \text{for } \chi \in S^{\text{CFE}},$$

we conclude that $T^{\text{CFE}} = (-\Delta^{\text{CFE}})^{-1}$. Notice that,

$$\rho = -(T^{\text{CFE}} - T)\Delta u = (R^{\text{CFE}} - I)u.$$

Set $e(t) = u^{\text{CFE}}(t) - u(t)$. For the problem (3.1), writing the semidiscrete problem and the continuous problem in terms of T^{CFE} and T, we find that

$$T^{\text{CFE}} u_t^{\text{CFE}} + u^{\text{CFE}} = 0 \quad \text{for} \;\; t > 0, \quad \text{with} \;\; u^{\text{CFE}}(0) = u_0^{\text{CFE}},$$

and

$$T u_t + u = 0 \quad \text{for} \;\; t > 0, \quad \text{with} \;\; u(0) = u_0.$$

Now, it is easy to derive the error equation as

$$\begin{aligned} T^{\text{CFE}} e_t + e &= (T^{\text{CFE}} u_t^{\text{CFE}} + u^{\text{CFE}}) - (T^{\text{CFE}} u_t + u) \\ &= (T - T^{\text{CFE}})\Delta u = \rho. \end{aligned}$$

Before we turn to the semidiscrete error estimate, we need some auxiliary results.

Lemma 3.2.1. *Let T^{CFE} be a selfadjoint and positive semidefinite on $L^2(\Omega)$ and*

$$T^{CFE} e_t + e = \rho \quad \textit{for} \;\; t \geq 0, \quad \textit{with} \;\; T^{CFE} e(0) = 0. \tag{3.6}$$

Then

$$\|e(t)\| \leq C \sup_{\zeta \leq t} (\zeta \|\rho_t(\zeta)\| + \|\rho(\zeta)\|) \quad \textit{for} \;\; t \geq 0.$$

Proof. Taking L^2-inner product of (3.6) with $2e_t$, we obtain

$$2(T^{\text{CFE}} e_t, e_t) + \frac{d}{dt}\|e\|^2 = 2(\rho, e_t),$$

which gives

$$\frac{d}{dt}\|e\|^2 \leq 2(\rho, e_t) = 2\frac{d}{dt}(\rho, e) - 2(\rho_t, e),$$

where we have used the fact that T^{CFE} is positive semidefinite. Now, multiplying the above by t, we have

$$\frac{d}{dt}(t\|e\|^2) \leq 2\frac{d}{dt}(t(\rho, e)) - 2t(\rho_t, e) + \|e\|^2 - 2(\rho, e).$$

Integrating over $(0, t)$, this yields

$$t\|e(t)\|^2 \leq 2t\|\rho\|\|e\| + \int_0^t (\|e\|^2 + 2\|\rho\|\|e\| + 2\zeta\|\rho_t\|\|e\|)d\zeta,$$

which implies

$$\|e(t)\|^2 \le C\left(\|\rho(t)\|^2 + \frac{1}{t}\int_0^t (\|e\|^2 + \|\rho\|^2 + \zeta^2\|\rho_t\|^2)d\zeta\right). \tag{3.7}$$

Now we need to estimate the term $\int_0^t \|e\|^2 d\zeta$. Taking the L^2-inner product of (3.6) by e and noting the fact that T^{CFE} is selfadjoint, we obtain

$$\frac{d}{dt}(T^{\mathrm{CFE}}e, e) + 2\|e\|^2 = 2(\rho, e) \le \|\rho\|^2 + \|e\|^2,$$

after integration

$$(T^{\mathrm{CFE}}e(t), e(t)) + \int_0^t \|e\|^2 d\zeta \le (T^{\mathrm{CFE}}e(0), e(0)) + \int_0^t \|\rho\|^2 d\zeta.$$

Since $T^{\mathrm{CFE}}e(0) = 0$ and T^{CFE} is positive semidefinite, we conclude that

$$\int_0^t \|e\|^2 d\zeta \le \int_0^t \|\rho\|^2 d\zeta,$$

which combine with (3.7) gives

$$\|e(t)\|^2 \le C\left(\|\rho(t)\|^2 + \frac{1}{t}\int_0^t (\|\rho\|^2 + \zeta^2\|\rho_t\|^2)d\zeta\right),$$

and hence the desired result follows. □

Remark 3.2.1. *By an obvious modification of the proof of Lemma 3.2.1, for $\epsilon > 0$ arbitrary, we obtain*

$$\|e(t)\|^2 \le \frac{\epsilon^2}{t}\int_0^t \zeta^2\|\rho_t\|^2 d\zeta + C_\epsilon\left(\|\rho(t)\|^2 + \frac{1}{t}\int_0^t \|\rho\|^2 d\zeta\right),$$

and this yields

$$\|e(t)\| \le \epsilon \sup_{\zeta \le t}(\zeta\|\rho_t(\zeta)\|) + C_\epsilon \sup_{\zeta \le t}\|\rho(\zeta)\| \quad \text{for } t \ge 0. \tag{3.8}$$

We now concentrate on the main auxiliary result which will be used in the semidiscrete error estimate.

Lemma 3.2.2. *Let the assumptions of Lemma 3.2.1 hold true. Then, we have*

$$\|e(t)\| \le Ct^{-1} \sup_{\zeta \le t}(\zeta^2\|\rho_t(\zeta)\| + \zeta\|\rho(\zeta)\| + \|\hat{\rho}(\zeta)\|) \quad \text{for } t > 0,$$

where $\hat{\rho}(t) = \int_0^t \rho(\zeta)d\zeta$.

Proof. Setting $\mathcal{V} = te$ and using (3.6) we have

$$T^{\mathrm{CFE}}\mathcal{V}_t + \mathcal{V} = \widetilde{\eta},$$

where $\widetilde{\eta} = t\rho + T^{\mathrm{CFE}}e$. By (3.8), we obtain

$$\|\mathcal{V}(t)\| \leq \epsilon \sup_{\zeta\leq t}(\zeta\|\widetilde{\eta}_t(\zeta)\|) + C_\epsilon \sup_{\zeta\leq t}\|\widetilde{\eta}(\zeta)\|. \tag{3.9}$$

Since

$$\widetilde{\eta}(\zeta) \leq \zeta\|\rho(\zeta)\| + \|T^{\mathrm{CFE}}e(\zeta)\|,$$

it now follows that

$$\begin{aligned}
\zeta\|\widetilde{\eta}_t(\zeta)\| &\leq \zeta\|\rho(\zeta)\| + \zeta^2\|\rho_t(\zeta)\| + \zeta\|T^{\mathrm{CFE}}e_t(\zeta)\| \\
&\leq \zeta^2\|\rho_t(\zeta)\| + 2\zeta\|\rho(\zeta)\| + \zeta\|e(\zeta)\| \\
&= \zeta^2\|\rho_t(\zeta)\| + 2\zeta\|\rho(\zeta)\| + \|\mathcal{V}(\zeta)\|,
\end{aligned}$$

where we have used the error equation (3.6) to obtain the second inequality. Therefore, with $\epsilon = 1/2$ we obtain from (3.9),

$$\|\mathcal{V}(t)\| \leq \frac{1}{2}\sup_{\zeta\leq t}\|\mathcal{V}(\zeta)\| + C\sup_{\zeta\leq t}(\zeta^2\|\rho_t(\zeta)\| + \zeta\|\rho(\zeta)\| + \|T^{\mathrm{CFE}}e(\zeta)\|).$$

Set $\sup_{\zeta\leq t}\|\mathcal{V}(\zeta)\| = \|\mathcal{V}(\xi^*)\|$ for some $\xi^*(t)$. Then

$$\|\mathcal{V}(t)\| \leq \|\mathcal{V}(\xi^*)\| \leq C\sup_{\zeta\leq t}(\zeta^2\|\rho_t(\zeta)\| + \zeta\|\rho(\zeta)\| + \|T^{\mathrm{CFE}}e(\zeta)\|). \tag{3.10}$$

To estimate $\|T^{\mathrm{CFE}}e\|$, we integrate the error equation (3.6) from 0 to t with $T^{\mathrm{CFE}}e(0) = 0$ to obtain

$$T^{\mathrm{CFE}}e + \hat{e} = T^{\mathrm{CFE}}\hat{e}_t + \hat{e} = \hat{\rho}, \tag{3.11}$$

where $\hat{e}(t) = \int_0^t e(\zeta)d\zeta$. By Lemma 3.2.1, we have

$$\|\hat{e}(t)\| \leq C\sup_{\zeta\leq t}(\zeta\|\hat{\rho}_t(\zeta)\| + \|\hat{\rho}(\zeta)\|) \leq C\sup_{\zeta\leq t}(\zeta\|\rho(\zeta)\| + \|\hat{\rho}(\zeta)\|).$$

Hence, (3.11) yields

$$\|T^{\mathrm{CFE}}e(t)\| \leq \|\hat{e}(t)\| + \|\hat{\rho}(t)\| \leq C\sup_{\zeta\leq t}(\zeta\|\rho(\zeta)\| + \|\hat{\rho}(\zeta)\|),$$

which together with (3.10) leads to

$$\|\mathcal{V}(t)\| \leq C\sup_{\zeta\leq t}(\zeta^2\|\rho_t\| + \zeta\|\rho\| + \|\hat{\rho}\|),$$

and this completes the rest of the proof. □

Now, we are in a position to derive the semidiscrete error estimate for nonsmooth initial data.

Theorem 3.2.1. *Let $u^{CFE}(t)$ and $u(t)$ be the solutions of (3.2) and (3.1), respectively and $u(t) \in H_0^1(\Omega) \cap H^2(\Omega)$ for each t. Assume that $u_0^{CFE} = P^{CFE}u_0$, where P^{CFE} denotes the L^2-projection onto S^{CFE}, and the conditions* (a) *and* (b) *hold. Then, we have*

$$\|u^{CFE}(t) - u(t)\| \leq CH^2\widetilde{Log}(H/h)\, t^{-1}\|u_0\| \quad \textit{for } \ t > 0.$$

Proof. By Lemma 3.2.2 we have

$$\|u^{\text{CFE}}(t) - u(t)\| \leq Ct^{-1}\sup_{\zeta \leq t}(\zeta^2\|\rho_t(\zeta)\| + \zeta\|\rho(\zeta)\| + \|\hat{\rho}(\zeta)\|) \quad \text{for } \ t > 0. \tag{3.12}$$

Using the condition (a) and equation (3.4), we have

$$\zeta\|\rho(\zeta)\| = \zeta\|(T^{\text{CFE}} - T)u_t(\zeta)\| \leq CH^2\widetilde{\text{Log}}(H/h)\zeta\|u_t(\zeta)\| \leq CH^2\widetilde{\text{Log}}(H/h)\|u_0\|.$$

Similarly,

$$\begin{aligned}\|\hat{\rho}(\zeta)\| &= \|\int_0^\zeta \rho(\tilde{\zeta})d\tilde{\zeta}\| = \|\int_0^\zeta (T^{\text{CFE}} - T)u_t\, d\tilde{\zeta}\| = \|(T^{\text{CFE}} - T)(u(\zeta) - u_0)\| \\ &\leq CH^2\widetilde{\text{Log}}(H/h)(\|u(\zeta)\| + \|u_0\|) \leq CH^2\widetilde{\text{Log}}(H/h)\|u_0\|,\end{aligned}$$

and

$$\begin{aligned}\zeta^2\|\rho_t(\zeta)\| &= \zeta^2\|(T^{\text{CFE}} - T)u_{tt}(\zeta)\| \\ &\leq CH^2\widetilde{\text{Log}}(H/h)\zeta^2\|u_{tt}(\zeta)\| \leq CH^2\widetilde{\text{Log}}(H/h)\|u_0\|.\end{aligned}$$

Combining the above inequalities with (3.12) we complete the rest of the proof. □

3.2.2 Fully Discrete Error Estimates

The spatially semidiscrete problem of (3.1) may be stated as: Find $u^{\text{CFE}}(t) \in S^{\text{CFE}}$ such that

$$u_t^{\text{CFE}} = \Delta^{\text{CFE}}u^{\text{CFE}} \quad \text{for } \ t > 0, \quad \text{with } \ u^{\text{CFE}}(0) = u_0^{\text{CFE}}, \tag{3.13}$$

where Δ^{CFE} is defined in (3.5). The equation (3.13) can be expressed in the form

$$u_t^{\text{CFE}} + \mathcal{B}u^{\text{CFE}} = 0 \quad \text{for } \ t > 0, \quad \text{with } \ u^{\text{CFE}}(0) = u_0^{\text{CFE}}, \tag{3.14}$$

where $\mathcal{B} = -\Delta^{\text{CFE}}$ is a linear, selfadjoint, positive definite operator in S^{CFE}.

With the solution operator $E(t)$, we can represent the exact solution $u(t)$ as in (3.3). Let $E^{\text{CFE}}(t)$ be the corresponding discrete analogue of the operator $E(t)$, so that $u^{\text{CFE}}(t)$ can be represented in terms of $E^{\text{CFE}}(t)$. Let $\{\lambda_j\}_{j=1}^{N^{\text{CFE}}}$ be the eigenvalues of operator $\mathcal{B}$ with the corresponding basis of orthonormal eigenfunctions $\{\varphi_j\}_{j=1}^{N^{\text{CFE}}}$. The spectrum of $\mathcal{B}$ be denoted by $\sigma(\mathcal{B})$ which is given by $\sigma(\mathcal{B}) = \{\lambda_j\}_{j=1}^{N^{\text{CFE}}}$. The solution operator $E^{\text{CFE}}(t)$ of (3.14) can be viewed as the exponential $e^{-t\mathcal{B}}$ (cf. [102]).

The backward Euler scheme is then given by

$$
\begin{aligned}
(U^n, \chi) + k\mathcal{A}(U^n, \chi) &= (U^{n-1}, \chi) \quad \forall \chi \in S^{\text{CFE}},\ n \geq 1, \\
\text{with } U^0 &= u_0^{\text{CFE}},
\end{aligned}
\tag{3.15}
$$

which can be written using (3.5) in the operator form as

$$(I - k\Delta^{\text{CFE}})U^n = U^{n-1},$$

or,

$$U^n = (I - k\Delta^{\text{CFE}})^{-1} U^{n-1}.$$

Let $r(\lambda)$ be a rational function approximating $e^{-\lambda}$. Hence for $n \geq 1$, U^n is recursively defined as

$$U^n = E_k U^{n-1} \quad \text{for } n \geq 1, \quad \text{with } U^0 = u_0^{\text{CFE}}, \tag{3.16}$$

where $E_k = r(k\mathcal{B})$ and the rational function $r(\lambda)$ is defined on the spectrum $\sigma(k\mathcal{B})$. Here, $r(\lambda) = \frac{1}{1+\lambda}$. Note that, $|r(\lambda)| \leq 1$ for $\lambda \geq 0$.

If the the scheme (3.16) is accurate of order 1, then we have (cf. [102])

$$r(\lambda) = e^{-\lambda} + \mathcal{O}(\lambda^2), \quad \text{as } \lambda \to 0. \tag{3.17}$$

Therefore, for λ_0 is small enough

$$|r(\lambda) - e^{-\lambda}| \leq C\lambda^2 \quad \text{for } 0 \leq \lambda \leq \lambda_0. \tag{3.18}$$

We can further restrict λ_0 such that

$$|r(\lambda)| \leq e^{-c\lambda} \quad \text{for } 0 \leq \lambda \leq \lambda_0, \quad \text{with } 0 < c < 1. \tag{3.19}$$

Classification of $r(\lambda)$. In order to estimate the error we will require some properties of the rational function $r(\lambda)$ approximating $e^{-\lambda}$. We classify $r(\lambda)$ of the following types for discretizations in time as follows:

(i) $|r(\lambda)| < 1$ for $0 < \lambda < \widetilde{\alpha}$, $\widetilde{\alpha} > 0$,

(ii) $|r(\lambda)| < 1$ for $\lambda > 0$,

(iii) $|r(\lambda)| < 1$ for $\lambda > 0$, and $|r(\infty)| < 1$.

For the purpose of application to the spatially semidiscrete problem (3.14), we further classify the schemes of the type (i) and (ii) as follows:

(i)′ $r(\lambda)$ is of type (i) and $k\lambda_{\max} \le \widetilde{\alpha}_0$, for some $\widetilde{\alpha}_0$ with $0 < \widetilde{\alpha}_0 < \widetilde{\alpha}$, and

(ii)′ $r(\lambda)$ is of type (ii) and $k\lambda_{\max} \le \widetilde{\alpha}_1$, for some $\widetilde{\alpha}_1$ with $0 < \widetilde{\alpha}_1 < \infty$,

where $\lambda_{\max}$ denotes the largest eigenvalue of $\mathcal{B} = -\Delta^{\mathrm{CFE}}$. For the schemes of type (i)′ and (ii)′ satisfying (3.17), we note that with $\lambda_0 = \widetilde{\alpha}_0$ and $\widetilde{\alpha}_1$, respectively, we obtain $|r(\lambda)| < 1$ for $0 < \lambda \le \lambda_0$, and thus (3.19) holds. Since $k\lambda \le \lambda_0$ for $\lambda \in \sigma(\mathcal{B})$, we have $|r(\lambda)| \le e^{-c\lambda}$ for $\lambda \in \sigma(k\mathcal{B})$ and $0 < c < 1$.

We now have the following result.

Lemma 3.2.3. *Let U^n and $u^{CFE}(t_n)$ be the solutions of (3.16) and (3.14), respectively. Assume that the discretization scheme is accurate of $\mathcal{O}(1)$ and of type* (i)′*,* (ii)′*, or* (iii)*. Then, we have*

$$\|U^n - u^{CFE}(t_n)\| \le Ckt_n^{-1}\|u_0^{CFE}\| \quad \textit{for } t_n > 0,$$

where the constant C depends on the parameters $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_1$ for type (i)′ *and* (ii)′*, respectively, and in case* (iii)*, C is independent of $\mathcal{B}$.*

Proof. To begin with, we first introduce the function $G_n(\lambda) = r(\lambda)^n - e^{-n\lambda}$. In view of (3.16), we write

$$U^n - u^{\mathrm{CFE}}(t_n) = r(k\mathcal{B})^n u_0^{\mathrm{CFE}} - e^{-nk\mathcal{B}} u_0^{\mathrm{CFE}} = G_n(k\mathcal{B}) u_0^{\mathrm{CFE}}.$$

To prove the result, we need to show that

$$\|G_n(k\mathcal{B}) u_0^{\mathrm{CFE}}\| \le Ckt_n^{-1}\|u_0^{\mathrm{CFE}}\|,$$

which can be written in terms of operator norm as

$$\|G_n(k\mathcal{B})\| \le Ckt_n^{-1} \quad \text{for } t_n > 0.$$

Since $t_n = nk$, it is enough to prove that

$$|G_n(\lambda)| \le Ckt_n^{-1} = Cn^{-1} \quad \text{for } \lambda \in \sigma(k\mathcal{B}),\ n \ge 1. \tag{3.20}$$

Note that for the schemes of type (i)′ and (ii)′, (3.19) is valid with $\lambda_0 = \widetilde{\alpha}_0$ and $\widetilde{\alpha}_1$, respectively. Further for schemes of type (iii), (3.19) is valid for any $\lambda_0 > 0$. Therefore using (3.18) together with (3.19), we have

$$\begin{aligned} |G_n(\lambda)| &= |r(\lambda)^n - e^{-n\lambda}| \\ &= |(r(\lambda) - e^{-\lambda})\sum_{j=0}^{n-1} r(\lambda)^{n-1-j}e^{-j\lambda}| \\ &= |r(\lambda) - e^{-\lambda}||\sum_{j=0}^{n-1} r(\lambda)^{n-1-j}e^{-j\lambda}| \\ &\leq Cn\lambda^2 e^{-cn\lambda} \\ &= Cn^{-1}(n\lambda)^2 e^{-cn\lambda} \leq Cn^{-1} \quad \text{for } \ 0 \leq \lambda \leq \lambda_0. \end{aligned}$$

Since $k\lambda_{\max} \leq \lambda_0$, the proof of (3.20) for the cases of types (i)′ and (ii)′ follows.

Now it remains to prove the lemma for the scheme of type (iii). For type (iii), we consider λ is large. Note that, (3.19) holds for λ_0 an arbitrary positive number. Therefore, for $\lambda \geq \lambda_0 = 1$, we have $e^{-n\lambda} \leq e^{-n} \leq Cn^{-1}$. Again, for $|r(\infty)| < 1$, we have for $c > 0$, $\sup_{\lambda \geq 1}|r(\lambda)| = e^{-c}$. This gives

$$\sup_{\lambda\geq 1}|r(\lambda)^n| \leq e^{-cn} \leq Cn^{-1}.$$

Therefore,

$$\sup_{\lambda\geq 1}|G_n(\lambda)| = \sup_{\lambda\geq 1}|r(\lambda)^n - e^{-n\lambda}| \leq Cn^{-1},$$

and this completes the proof. □

Now we are in a position to show the error estimate for nonsmooth initial data for the backward Euler CFE method.

Theorem 3.2.2 (Backward Euler method)**.** *Let U^n and $u(t_n)$ be the solutions of (3.15) and (3.1), respectively and $u \in H_0^1(\Omega) \cap H^2(\Omega)$. Then under the assumptions of Lemma 3.2.3 with $u_0^{CFE} = P^{CFE}u_0$, where P^{CFE} is the L^2-projection onto S^{CFE}, we have*

$$\|U^n - u(t_n)\| \leq C(H^2\widetilde{Log}(H/h)t_n^{-1} + kt_n^{-1})\|u_0\| \quad \textit{for } \ t_n > 0.$$

Proof. By the triangle inequality we write the error as

$$\|U^n - u(t_n)\| \leq \|U^n - u^{\text{CFE}}(t_n)\| + \|u^{\text{CFE}}(t_n) - u(t_n)\|. \tag{3.21}$$

For the first term, we have from Lemma 3.2.3 that

$$\|U^n - u^{\text{CFE}}(t_n)\| \leq Ckt_n^{-1}\|P^{\text{CFE}}u_0\| \leq Ckt_n^{-1}\|u_0\| \quad \text{for } \ t_n > 0,$$

and, by Theorem 3.2.1 the second term on the right of (3.21) gives

$$\|u^{\mathrm{CFE}}(t_n) - u(t_n)\| \leq CH^2 \widetilde{\mathrm{Log}}(H/h) t_n^{-1} \|u_0\| \quad \text{for} \;\; t_n > 0,$$

which completes the proof. □

3.3 Concluding Remarks

In this chapter, we have considered a two-scale CFEs for the discretization of homogeneous parabolic equation in convex domains with nonsmooth initial data. We use standard piecewise linear and continuous CFEs for the space discretization and derive *a priori* error estimates for the semi-discretization problem, while the backward Euler method is then used for time discretization and *a priori* error estimates for the fully discrete problem are also derived. Further, for nonsmooth initial data, optimal order estimate (up to logarithmic terms) for positive time is proved in the $L^\infty(L^2)$-norm. The proposed CFE method enables us to provide a powerful technique to resolve the Dirichlet boundary conditions without increasing the degrees of freedom. The spatially semidiscrete and fully discrete error bounds (Theorems 3.2.1 and 3.2.2) do not pose any restriction on the fine-scale parameter h, and in the case $h = \mathcal{O}(H)$, the results are coincide with the result of standard FEM for convex domains. Numerical results for a test problem are presented in Chapter 7 (Example 7.2) to support our theoretical analysis. A comparison with standard FEM in terms of the $\sharp$dof is also given in Chapter 7 (see Tables 7.11 and 7.12).

4

Two-Scale CFE Method for Parabolic Problems in Nonconvex Polygonal Domains

This chapter is devoted to the *a priori* error analysis for the spatially semidiscrete and fully discrete CFE approximations of the problem (1.1) in a nonconvex polygonal domain for both smooth and nonsmooth initial data. The discretization with respect to space is by piecewise linear CFEs, and in time we have applied the backward Euler approximation. We derive the spatially semidiscrete and fully discrete error estimates in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms with smooth initial data. In addition, for homogeneous equation, error estimates for the spatially semidiscrete and fully discrete methods are established in the $L^\infty(L^2)$-norm for nonsmooth initial data.

4.1 Introduction

We shall begin by rewriting the model IBVP for the heat equation

$$\begin{aligned} u_t - \Delta u &= f \quad \text{in } \Omega, \; t > 0, \\ u &= 0 \quad \text{on } \Gamma, \; t > 0, \\ u(\cdot, 0) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{4.1}$$

where Ω is a nonconvex polygonal domain in $\mathbb{R}^2$ with boundary Γ. Here $u = u(x,t)$, u_t denotes $\partial u/\partial t$, and the Laplacian is denoted by $\Delta = \sum_{j=1}^{2} \partial^2/\partial x_j^2$.

To present the CFE method for the problem (4.1), we now introduce its weak formulation: For each $t > 0$, find $u(t) \in H_0^1(\Omega)$ such that

$$\begin{aligned} (u_t(t), \varphi) + \mathcal{A}(u(t), \varphi) &= (f(t), \varphi) \quad \forall \varphi \in H_0^1(\Omega), \; t > 0, \\ u(0) &= u_0(x), \end{aligned} \tag{4.2}$$

where the bilinear form $\mathcal{A}(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$\mathcal{A}(v,w) = (\nabla v, \nabla w) \quad \forall v, w \in H_0^1(\Omega).$$

The bilinear form $\mathcal{A}(\cdot,\cdot)$ is bounded and coercive on $H_0^1(\Omega)$, i.e., $\exists\, \alpha_0,\ \beta_0 > 0$ such that

$$|\mathcal{A}(v,w)| \le \alpha_0 \|v\|_1 \|w\|_1 \quad \forall v, w \in H_0^1(\Omega),$$

and

$$\mathcal{A}(v,v) \ge \beta_0 \|v\|_1^2 \quad \forall v \in H_0^1(\Omega).$$

For the purpose of CFE approximations, we need to define the CFE space. Let S be the continuous and piecewise linear finite element space corresponding to the triangulation $\mathcal{T}_{H,h}$ is defined by

$$S := \{v \in C^0(\Omega_{H,h}) \ \mid \ v|_K \in \mathbb{P}_1 \ \ \forall K \in \mathcal{T}_{H,h}\},$$

where $\Omega_{H,h} := \operatorname{int}\left(\bigcup_{K\in\mathcal{T}_{H,h}} K\right)$. Then the CFE space S^{CFE} (as described in Chapter 1) on the two-scale mesh $\mathcal{T}_{H,h}$ is now defined by

$$S^{\mathrm{CFE}} := \{v \in S \mid \exists\, \mathbf{\Phi} \in \mathbb{R}^{\vartheta_{\mathrm{dof}}} \ \text{ with } v(x) = (\mathcal{E}\mathbf{\Phi})_x \ \ \forall x \in \vartheta_{H,h}\},$$

where $\mathbf{\Phi} : \vartheta_{\mathrm{dof}} \to \mathbb{R}$ is the grid function and the extrapolation operator $\mathcal{E} : \mathbb{R}^{\vartheta_{\mathrm{dof}}} \to \mathbb{R}^{\vartheta_{H,h}}$ is defined in (1.11). From the definition of S^{CFE} it is clear that $S^{\mathrm{CFE}} \subset S$, where the values at the slave nodes are restricted by the extrapolation. The CFE approximation is now stated as follows: Find $u^{\mathrm{CFE}}(t) := u^{\mathrm{CFE}}(\cdot,t) \in S^{\mathrm{CFE}}$ such that

$$\begin{aligned} (u_t^{\mathrm{CFE}}, \chi) + \mathcal{A}(u^{\mathrm{CFE}}, \chi) \ &= \ (f,\chi) \quad \forall \chi \in S^{\mathrm{CFE}},\ t > 0, \\ u^{\mathrm{CFE}}(0) \ &= \ u_0^{\mathrm{CFE}}, \end{aligned} \tag{4.3}$$

where u_0^{CFE} is some suitable approximation of u_0 in S^{CFE}.

The fully discrete backward Euler scheme for both constant and variable time step are discussed. We shall use the following notations for the fully discretizations scheme: Let k be the constant time step with $t = t_n = nk$, and U^n be the approximation of u^n in S^{CFE}, where $u^n = u(t_n)$. We shall use the backward Euler quotient for the constant time step

$$\bar{\partial} U^n := (U^n - U^{n-1})/k.$$

For the backward Euler scheme with variable time step, let $0 = t_0 < t_1 < \cdots < t_n < \cdots$ be a partition of the positive time axis with the variable time step $k_n = t_n - t_{n-1}$. Then, we define the backward Euler quotient for the variable time step as

$$\bar{\partial}_n U^n := (U^n - U^{n-1})/k_n.$$

FEM for linear parabolic PDEs in nonconvex polygonal domains were studied by Chatzipantelidis *et al.* [22]. They have considered piecewise linear finite elements with respect to space discretization. The regularity of the solutions of corresponding stationary elliptic problem has been extensively studied by Bacuta *et al.* [8] and Grisvard [47, 48]. If the physical domain is nonconvex or very complicated, the standard FEMs require to generate finite element mesh that resolves the domain boundary. The minimal number of simplices can be very large and distributed in a nonoptimal way with respect to the approximation quality. As a result, the dimension of the finite element space becomes very large which increases the computational effort. The authors of [22] have derived *a priori* error bounds in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms and shown that the order of convergence for the nonconvex domains with respect to spatial discretization is reduced from the optimal order $\mathcal{O}(h^2)$ (the order of convergence in convex polygonal domains) because of the lack of regularity of the exact solution caused by the re-entrant corner. However, they have introduced a suitable mesh refinements toward the nonconvex corner which essentially restore the order $\mathcal{O}(h^2)$ error bound (cf. Schatz and Wahlbin [94]). Rech *et al.* [86] have investigated the two-scale CFE method for the Poisson equation with homogeneous Dirichlet boundary condition in two-dimensional nonconvex domains. The authors have demonstrated the convergence of order $\mathcal{O}(H^{1+s})$, $1/2 \leq s \leq 1$ in the L^2-norm and $\mathcal{O}(H^s \widetilde{\mathrm{Log}}^{s/2}(H/h))$, $1/2 \leq s \leq 1$ in the H^1-norm.

Since elliptic error estimates play a crucial role in the error analysis for parabolic problems, an attempt has been made in this chapter to generalize two-scale CFE method for elliptic problem to parabolic problem in nonconvex polygonal domains with smooth and nonsmooth initial data. We have studied both semidiscrete and fully discrete CFE methods and derive convergence properties in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms. Further, for the homogeneous problem, error estimates are derived when the given initial data u_0 is only in $L^2(\Omega)$.

The rest of this chapter is planned as follows. *A priori* error estimates for the spatially semidiscrete and fully discrete CFE methods with smooth initial data are carried out in Section 4.2. In Section 4.3, *a priori* error estimates for nonsmooth initial data are discussed. Finally, some concluding remarks are given in Section 4.4.

4.2 CFE Error Estimates

This section is concerned with both the spatially semidiscrete and fully discrete CFE error analysis for smooth initial data. The convergence properties for the CFE solutions in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms are established.

4.2.1 Spatially Semidiscrete Error Estimates

For the purpose of semidiscrete error analysis we recall the *elliptic* or *Ritz projection* $R^{\text{CFE}} : H_0^1(\Omega) \to S^{\text{CFE}}$ be defined by

$$(\nabla R^{\text{CFE}} v, \nabla \chi) = (\nabla v, \nabla \chi) \quad \text{for } v \in H_0^1(\Omega),\ \forall \chi \in S^{\text{CFE}}. \tag{4.4}$$

$R^{\text{CFE}} v$ satisfies the following stability estimate:

$$\|\nabla R^{\text{CFE}} v\| \le \|\nabla v\| \quad \forall v \in H_0^1(\Omega).$$

Since Ω is nonconvex, the exact solution $u \notin H^2(\Omega) \cap H_0^1(\Omega)$, that is, the exact solution u of the problem (4.1) have a lower regularity due to the re-entrant corner of the boundary Γ. Therefore, for the nonconvex domain $u \in H_0^1(\Omega) \cap H^{1+s}(\Omega)$, $1/2 \le s \le 1$ (cf. [86]). For the regularity of elliptic problems in nonconvex domains and problems having a singularity in the solutions, we refer to [8, 47, 48].

The next lemma gives an estimate for $(R^{\text{CFE}} u - u)$ in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms.

Lemma 4.2.1. *Let $u \in H_0^1(\Omega) \cap H^{1+s}(\Omega)$, $1/2 \le s \le 1$ and $R^{CFE} : H_0^1(\Omega) \to S^{CFE}$ be defined by (4.4). Assume that the conditions (1.9) and (2.9) hold true. Then, we have*

$$\|\nabla (R^{CFE} u - u)\| \le C H^s \widetilde{Log}^{s/2}(H/h) \|u\|_{1+s,\Omega} \quad \textit{for } 1/2 \le s \le 1, \tag{4.5}$$

and

$$\|R^{CFE} u - u\| \le C H^{2s} \widetilde{Log}^{s}(H/h) \|u\|_{1+s,\Omega} \quad \textit{for } 1/2 \le s \le 1, \tag{4.6}$$

where the constant C depends on σ_{dist}, σ_e, σ_{uni}, $C_{ol,1}$, $C_{ol,2}$, σ_{ext} and the minimal angles in the triangulation $\mathcal{T}_{H,h}$ (as described in Chapter 2).

Proof. We shall follow the tricks from [16] and [86, Theorem 2]. Let $\mathcal{L}$ be an operator that maps u to the error $u - R^{\text{CFE}} u$ is defined by $\mathcal{L} u := u - R^{\text{CFE}} u$. Clearly, the operator $\mathcal{L}$ is linear as the projection is a linear operation on Hilbert spaces.

It follows from [86, Theorem 1] and Lemma 2.4.1 that $\mathcal{L}$ maps $H^2(\Omega)$ to $H^1(\Omega)$ and the estimates

$$\|\mathcal{L}\|_{H^2(\Omega) \to H^1(\Omega)} \le C H \widetilde{\text{Log}}^{1/2}(H/h)$$

hold.

Note that, $\mathcal{L}$ also maps $H^1(\Omega)$ to $H^1(\Omega)$. In view of $\|u - R^{\text{CFE}} u\|_{1,\Omega} \le \|u\|_{1,\Omega}$, we have

$$\|\mathcal{L}\|_{H^1(\Omega) \to H^1(\Omega)} \le 1.$$

Further, for $1/2 \le s \le 1$, $\mathcal{L}$ maps $H^{1+s}(\Omega)$ to $H^1(\Omega)$ (cf. [86, Lemma 3]), and this yields

$$\|\mathcal{L}\|_{H^{1+s}(\Omega)\to H^1(\Omega)} \le CH^s\widetilde{\mathrm{Log}}^{s/2}(H/h),$$

which completes the proof of (4.5). In order to show (4.6), we use the duality argument. Let $\psi \in H_0^1(\Omega) \cap H^{1+s}(\Omega)$, $1/2 \le s \le 1$ be the solution of

$$-\Delta\psi = \varphi \quad \text{in } \ \Omega, \qquad \text{with} \quad \psi = 0 \quad \text{on } \ \Gamma,$$

where $\varphi \in L^2(\Omega)$. With $\chi = R^{\mathrm{CFE}}\psi$, we have

$$\begin{aligned}
(R^{\mathrm{CFE}}u - u, \varphi) &= (\nabla(R^{\mathrm{CFE}}u - u), \nabla\psi) \\
&= (\nabla(R^{\mathrm{CFE}}u - u), \nabla(\psi - \chi)) \\
&\le \|\nabla(R^{\mathrm{CFE}}u - u)\|\|\nabla(\psi - \chi)\| \\
&\le (CH^s\widetilde{\mathrm{Log}}^{s/2}(H/h)\|u\|_{1+s})(CH^s\widetilde{\mathrm{Log}}^{s/2}(H/h)\|\psi\|_{1+s}) \\
&= CH^{2s}\widetilde{\mathrm{Log}}^{s}(H/h)\|u\|_{1+s}\|\psi\|_{1+s},
\end{aligned}$$

using the regularity estimate $\|\psi\|_{1+s} \le C\|\Delta\psi\| = C\|\varphi\|$ (see [8, 22]), we have

$$(R^{\mathrm{CFE}}u - u, \varphi) \le CH^{2s}\widetilde{\mathrm{Log}}^{s}(H/h)\|u\|_{1+s}\|\|\varphi\|,$$

which implies

$$\|R^{\mathrm{CFE}}u - u\| \le CH^{2s}\widetilde{\mathrm{Log}}^{s}(H/h)\|u\|_{1+s,\Omega}.$$

This completes the proof of the lemma. $\square$

Remark 4.2.1. *According to Lemma 4.2.1 one can easily obtain the temporal derivative of* $\nabla(R^{CFE}u - u)$ *and* $(R^{CFE}u - u)$, *for* $1/2 \le s \le 1$, *as*

$$\|\nabla(R^{CFE}u_t - u_t)\| \le CH^s\widetilde{Log}^{s/2}(H/h)\|u_t\|_{1+s,\Omega},$$

and

$$\|R^{CFE}u_t - u_t\| \le CH^{2s}\widetilde{Log}^{s}(H/h)\|u_t\|_{1+s,\Omega}. \tag{4.7}$$

The following theorem presents the semidiscrete CFE error estimates.

Theorem 4.2.1. *Let* $u \in H_0^1(\Omega) \cap H^{1+s}(\Omega)$, $1/2 \le s \le 1$, *be the solution of (4.1) and let* u^{CFE} *be the solution of (4.3). Assume that the conditions (1.9) and (2.9) hold* *true. Then there exists a positive constant C independent of (h,H) such that, for $t \geq 0$, $1/2 \leq s \leq 1$, we have*

$$\|u^{CFE}(t) - u(t)\| \leq \|u_0^{CFE} - u_0\| \tag{4.8}$$
$$+ CH^{2s}\widetilde{Log}^{s}(H/h)\left[\|u_0\|_{1+s,\Omega} + \int_0^t \|u_t\|_{1+s,\Omega} d\zeta\right],$$

and

$$\|\nabla(u^{CFE}(t) - u(t))\| \leq C\|\nabla(u_0^{CFE} - u_0)\| + CH^{s}\widetilde{Log}^{s/2}(H/h)\Big[\|u_0\|_{1+s,\Omega} \tag{4.9}$$
$$+ \|u(t)\|_{1+s,\Omega} + (\int_0^t \|u_t\|^2_{1+s,\Omega} d\zeta)^{1/2}\Big].$$

Proof. We first decompose the error $(u^{\text{CFE}}(t) - u(t))$ as

$$\begin{aligned} u^{\text{CFE}}(t) - u(t) &= (u^{\text{CFE}}(t) - R^{\text{CFE}}u(t)) + (R^{\text{CFE}}u(t) - u(t)) \\ &= \theta(t) + \rho(t). \end{aligned}$$

In view of Lemma 4.2.1, we have

$$\|\rho(t)\| \leq CH^{2s}\widetilde{\text{Log}}^{s}(H/h)\|u(t)\|_{1+s,\Omega}$$
$$\leq CH^{2s}\widetilde{\text{Log}}^{s}(H/h)(\|u_0\|_{1+s,\Omega} + \int_0^t \|u_t\|_{1+s,\Omega} d\zeta). \tag{4.10}$$

It now remains to estimate $\theta(t)$. Using the definition of Ritz projection (4.4), together with (4.2) and (4.3), we obtain

$$(\theta_t, \chi) + \mathcal{A}(\theta, \chi) = -(\rho_t, \chi) \quad \forall \chi \in S^{\text{CFE}},\ t > 0. \tag{4.11}$$

We now introduce a *discrete Laplacian* $\Delta^{\text{CFE}} : S^{\text{CFE}} \to S^{\text{CFE}}$ defined by

$$(-\Delta^{\text{CFE}}\psi, \chi) = (\nabla\psi, \nabla\chi) \quad \forall\, \psi, \chi \in S^{\text{CFE}}.$$

It can be easily verified that the operator $-\Delta^{\text{CFE}}$ is selfadjoint and positive definite in S^{CFE} with respect to $(\cdot,\cdot)$. Note that

$$(\Delta^{\text{CFE}}R^{\text{CFE}}v, \chi) = -(\nabla R^{\text{CFE}}v, \nabla\chi) = -(\nabla v, \nabla \chi)$$
$$= (\Delta v, \chi) = (P^{\text{CFE}}\Delta v, \chi) \quad \forall\, \chi \in S^{\text{CFE}},$$

where P^{CFE} denotes the orthogonal L^2-projection onto S^{CFE}. Thus,

$$\Delta^{\text{CFE}}R^{\text{CFE}} = P^{\text{CFE}}\Delta.$$

Now, we can express (4.11) as

$$(\theta_t, \chi) - (\Delta^{\mathrm{CFE}}\theta, \chi) = -(P^{\mathrm{CFE}}\rho_t, \chi) \quad \forall \chi \in S^{\mathrm{CFE}},\ t > 0,$$

which can be written as

$$\theta_t - \Delta^{\mathrm{CFE}}\theta = -P^{\mathrm{CFE}}\rho_t, \ \ t > 0, \qquad \text{with} \ \ \theta(0) = u_0^{\mathrm{CFE}} - R^{\mathrm{CFE}}u_0. \tag{4.12}$$

Let $E^{\mathrm{CFE}}(t) = e^{\Delta^{\mathrm{CFE}}t}$ be the discrete analogue of the solution operator $E(t)$ of the homogeneous case of (4.1). Then the solution of (4.12) is given by

$$\theta(t) = E^{\mathrm{CFE}}(t)\theta(0) - \int_0^t E^{\mathrm{CFE}}(t-\zeta)P^{\mathrm{CFE}}\rho_t(\zeta)d\zeta.$$

Since both $E^{\mathrm{CFE}}(t)$ and P^{CFE} are stable in the L^2-norm, we have

$$\begin{aligned}\|\theta(t)\| &\le \|\theta(0)\| + \int_0^t \|\rho_t\| d\zeta \\ &\le \|u_0^{\mathrm{CFE}} - u_0\| + \|\rho(0)\| + \int_0^t \|\rho_t\| d\zeta,\end{aligned}$$

which combine with (4.7) and (4.10), we obtain

$$\|\theta(t)\| \le \|u_0^{\mathrm{CFE}} - u_0\| + CH^{2s}\widetilde{\mathrm{Log}}^s(H/h)(\|u_0\|_{1+s,\Omega} + \int_0^t \|u_t\|_{1+s,\Omega} d\zeta),$$

and this proves the estimate (4.8).

Next, for the gradient error estimate (4.9), we write $\nabla(u^{\mathrm{CFE}}(t) - u(t)) = \nabla\theta(t) + \nabla\rho(t)$. Since $\nabla\rho(t)$ is bounded by Lemma 4.2.1, we need to estimate $\nabla\theta(t)$. Now, choose $\chi = \theta_t$ in (4.11) to obtain

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 = -(\rho_t, \theta_t) \le \frac{1}{2}\|\rho_t\|^2 + \frac{1}{2}\|\theta_t\|^2,$$

and hence $\frac{d}{dt}\|\nabla\theta\|^2 \le \|\rho_t\|^2$. Which upon integration from 0 to t yields

$$\begin{aligned}\|\nabla\theta(t)\|^2 &\le \|\nabla\theta(0)\|^2 + \int_0^t \|\rho_t\|^2 d\zeta \\ &\le (\|\nabla(u_0^{\mathrm{CFE}} - u_0)\| + \|\nabla(R^{\mathrm{CFE}}u_0 - u_0)\|)^2 + \int_0^t \|\rho_t\|^2 d\zeta.\end{aligned}$$

Now using (4.7) and Lemma 4.2.1, we obtain

$$\|\nabla\theta(t)\|^2 \le 2\|\nabla(u_0^{\mathrm{CFE}} - u_0)\|^2 + CH^{2s}\widetilde{\mathrm{Log}}^s(H/h)\left(\|u_0\|^2_{1+s,\Omega} + \int_0^t \|u_t\|^2_{1+s,\Omega} d\zeta\right),$$

which combine with the estimate for $\nabla\rho(t)$ completes the rest of the proof. □

Remark 4.2.2. *With $h = \mathcal{O}(H)$, the results of Theorem 4.2.1 coincide with the results of standard FEM for nonconvex domains. This is because in this case the two-scale grid $\mathcal{T}_{H,h}$ coincides with the coarse grid $\mathcal{T}_H$. Theorem 4.2.1 does not pose any restriction on the fine-scale parameter h.*

4.2.2 Fully Discrete Error Estimates

This section is devoted to the fully discrete CFE method based on the backward Euler time discretization scheme. Here, the spatially semidiscrete problem (4.3) is discretized with respect to time for both constant as well as variable time steps.

Backward Euler method with constant time step: Let k be the constant time step with $t_n = nk$, and U^n be the approximation of u^n in S^{CFE}, where $u^n = u(t_n)$.

The fully discrete backward Euler CFE method is defined by replacing the time derivative in (4.3) as follows: Find $U^n \in S^{\text{CFE}}$ such that

$$(\bar{\partial}U^n, \chi) + \mathcal{A}(U^n, \chi) = (f(t_n), \chi) \quad \forall \chi \in S^{\text{CFE}},\ n \geq 1, \qquad \text{with} \quad U^0 = u_0^{\text{CFE}}, \tag{4.13}$$

which gives

$$(U^n, \chi) + k\mathcal{A}(U^n, \chi) = (U^{n-1} + kf(t_n), \chi) \quad \forall \chi \in S^{\text{CFE}}.$$

Using the same notation as in the semidiscrete case (in Chapter 2), the above equation can be expressed in vector and matrix form as

$$(A^{\text{CFE}} + kB^{\text{CFE}})\alpha^n = A^{\text{CFE}}\alpha^{n-1} + kf^{\text{CFE}}(t_n),$$

where $A^{\text{CFE}} + kB^{\text{CFE}}$ is positive definite and hence, in particular, invertible.

We shall prove the following error estimate:

Theorem 4.2.2. *Let U^n and u be the solutions of (4.13) and (4.1), respectively. Then for $n \geq 0$, $1/2 \leq s \leq 1$, we have*

$$\|U^n - u(t_n)\| \leq \|u_0^{CFE} - u_0\| + CH^{2s}\widetilde{Log}^s(H/h)\left(\|u_0\|_{1+s,\Omega} + \int_0^{t_n} \|u_t\|_{1+s,\Omega} d\zeta\right) + k\int_0^{t_n} \|u_{tt}\| d\zeta.$$

Proof. Splitting the error term, we have

$$U^n - u(t_n) = (U^n - R^{\text{CFE}}u(t_n)) + (R^{\text{CFE}}u(t_n) - u(t_n)) = \theta^n + \rho^n.$$

Since the estimate of ρ^n is known from Lemma 4.2.1, we need only to bound θ^n. It is easy to verify that θ^n satisfies

$$(\bar{\partial}\theta^n, \chi) + \mathcal{A}(\theta^n, \chi) = -(\omega^n, \chi) \quad \forall \chi \in S^{\mathrm{CFE}},\ n \geq 1, \tag{4.14}$$

where

$$\omega^n = R^{\mathrm{CFE}}\bar{\partial}u(t_n) - u_t(t_n) = (R^{\mathrm{CFE}} - I)\bar{\partial}u(t_n) + (\bar{\partial}u(t_n) - u_t(t_n)) = \omega_1^n + \omega_2^n.$$

Choose $\chi = \theta^n$ in (4.14) to obtain

$$\|\theta^n\| \leq \|\theta^0\| + k\sum_{j=1}^{n}\|\omega_1^j\| + k\sum_{j=1}^{n}\|\omega_2^j\|. \tag{4.15}$$

The first term on the right-hand side of (4.15) yields

$$\begin{aligned}\|\theta^0\| \leq & \ \|u_0^{\mathrm{CFE}} - u_0\| + \|R^{\mathrm{CFE}}u_0 - u_0\| \\ \leq & \ \|u_0^{\mathrm{CFE}} - u_0\| + CH^{2s}\widetilde{\mathrm{Log}}^s(H/h)\|u_0\|_{1+s,\Omega}.\end{aligned}$$

For the second term, using the arguments as in [102] to obtain

$$\begin{aligned} k\sum_{j=1}^{n}\|\omega_1^j\| & \leq \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} CH^{2s}\widetilde{\mathrm{Log}}^s(H/h)\|u_t\|_{1+s,\Omega}d\zeta \\ & = CH^{2s}\widetilde{\mathrm{Log}}^s(H/h)\int_0^{t_n}\|u_t\|_{1+s,\Omega}d\zeta,\end{aligned}$$

and similarly for the third term,

$$k\sum_{j=1}^{n}\|\omega_2^j\| \leq \sum_{j=1}^{n}\|\int_{t_{j-1}}^{t_j}(\zeta - t_{j-1})u_{tt}(\zeta)d\zeta\| \leq k\int_0^{t_n}\|u_{tt}\|d\zeta.$$

Altogether these estimates complete the proof of the theorem. □

Remark 4.2.3. *In order to estimate the error for the gradient we split* $\nabla(U^n - u(t_n)) = \nabla\theta^n + \nabla\rho^n$*, where* $\nabla\rho^n$ *is bounded as desired by Lemma 4.2.1. Further, choose* $\chi = \bar{\partial}\theta^n$ *in (4.14) to obtain*

$$\bar{\partial}\|\nabla\theta^n\|^2 \leq \|\omega^n\|^2,$$

and with $\nabla\theta^0 = 0$*, we get*

$$\|\nabla\theta^n\|^2 \leq k\sum_{j=1}^{n}\|\omega^j\|^2 \leq Ck\sum_{j=1}^{n}\|\omega_1^j\|^2 + Ck\sum_{j=1}^{n}\|\omega_2^j\|^2.$$

Using the estimates for ω_1^j *and* ω_2^j *as before together with the estimate of* $\|\nabla\rho\|$*, we obtain*

$$\|\nabla(U^n - u(t_n))\| \leq C(u)(H^s\widetilde{Log}^{s/2}(H/h) + k) \quad \text{for } 1/2 \leq s \leq 1.$$

In the above time discretization schemes we have used a constant time step k. We shall now study fully discrete method of a variable time step for the backward Euler method.

Backward Euler method with variable time step: Let $0 = t_0 < t_1 < \cdots < t_n < \cdots$ be a partition of the positive time axis with the variable time step $k_n = t_n - t_{n-1}$, and let U^n be the approximation of $u(t_n)$. Set $\bar{\partial}_n U^n = (U^n - U^{n-1})/k_n$. Thus, (4.13) will be replaced by

$$(\bar{\partial}_n U^n, \chi) + \mathcal{A}(U^n, \chi) = (f(t_n), \chi) \quad \forall \chi \in S^{\text{CFE}},\ n \geq 1, \tag{4.16}$$
$$\text{with} \quad U^0 = u_0^{\text{CFE}}.$$

The following theorem shows the $L^\infty(L^2)$ error estimate with variable time steps.

Theorem 4.2.3. *Let U^n and u be the solutions of (4.16) and (4.1), respectively. Then, we have for $n \geq 0$, $1/2 \leq s \leq 1$,*

$$\|U^n - u(t_n)\| \leq \|u_0^{CFE} - u_0\| + CH^{2s}\widetilde{Log}^s(H/h)\Big(\|u_0\|_{1+s,\Omega} + \int_0^{t_n} \|u_t\|_{1+s,\Omega} d\zeta\Big) + \sum_{j=1}^n k_j \int_{t_{j-1}}^{t_j} \|u_{tt}\| d\zeta.$$

Proof. We first write $U^n - u^n = \theta^n + \rho^n$. This time θ^n satisfies

$$(\bar{\partial}_n \theta^n, \chi) + \mathcal{A}(\theta^n, \chi) = -(\omega^n, \chi) \quad \forall \chi \in S^{\text{CFE}},\ n \geq 1,$$

where

$$\omega^n = (R^{\text{CFE}} - I)\bar{\partial}_n u^n + (\bar{\partial}_n u^n - u_t^n) = \omega_1^n + \omega_2^n.$$

Analogous to (4.15), we obtain

$$\|\theta^n\| \leq \|\theta^0\| + \sum_{j=1}^n k_j \big(\|\omega_1^j\| + \|\omega_2^j\|\big).$$

As before, we estimate

$$\begin{aligned} \sum_{j=1}^n k_j \|\omega_1^j\| &\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} CH^{2s}\widetilde{\text{Log}}^s(H/h)\|u_t\|_{1+s,\Omega} d\zeta \\ &= CH^{2s}\widetilde{\text{Log}}^s(H/h) \int_0^{t_n} \|u_t\|_{1+s,\Omega} d\zeta, \end{aligned}$$

and

$$\sum_{j=1}^n k_j \|\omega_2^j\| \leq \sum_{j=1}^n \Big\| \int_{t_{j-1}}^{t_j} (\zeta - t_{j-1}) u_{tt}(\zeta) d\zeta \Big\| \leq \sum_{j=1}^n k_j \int_{t_{j-1}}^{t_j} \|u_{tt}\| d\zeta.$$

The above estimates together with the standard estimates for ρ^n and θ^0 completes the rest of the proof. □

4.3 Nonsmooth Data Error Estimates

This section considers *a priori* error analysis for the CFE method when $u_0 \in L^2(\Omega)$ and the convergence properties for the CFE solution for positive time is established.

We shall rewrite the homogeneous form of (4.1) as

$$u_t = \Delta u \quad \text{in } \Omega,\ t > 0, \qquad (4.17)$$
$$u = 0 \quad \text{on } \Gamma,\ t > 0, \quad \text{with} \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega,$$

where Ω is a nonconvex polygonal domain in $\mathbb{R}^2$ with boundary Γ and the initial data $u_0 \in L^2(\Omega)$. We now write the weak formulation of (4.17) as follows: For each $t > 0$, find $u(t) \in H_0^1(\Omega)$ such that

$$\begin{aligned} (u_t(t), \varphi) + \mathcal{A}(u(t), \varphi) &= 0 \quad \forall \varphi \in H_0^1(\Omega),\ t > 0, \\ u(0) &= u_0(x). \end{aligned}$$

The spatially semidiscrete CFE approximation of (4.17) is stated as follows: Find $u^{\text{CFE}}(t) \in S^{\text{CFE}}$ such that

$$\begin{aligned} (u_t^{\text{CFE}}, \chi) + \mathcal{A}(u^{\text{CFE}}, \chi) &= 0 \quad \forall \chi \in S^{\text{CFE}},\ t > 0, \\ u^{\text{CFE}}(0) &= u_0^{\text{CFE}}, \end{aligned} \qquad (4.18)$$

where u_0^{CFE} is some suitable approximation of u_0 in S^{CFE}.

We now introduce certain Hilbert spaces $\dot{H}^\nu(\Omega)$ (cf. [22]). Consider the orthonormal eigenfunctions $\{\varphi_j\}_{j=1}^\infty$ and the corresponding eigenvalues $\{\lambda_j\}_{j=1}^\infty$ associated with the eigenvalue problem

$$-\Delta\varphi = \lambda\varphi \quad \text{in } \Omega, \quad \text{with} \quad \varphi = 0 \quad \text{on } \Gamma.$$

Since $\{\varphi_j\}_{j=1}^\infty$ form an orthonormal basis in $L^2(\Omega)$, for each $v \in L^2(\Omega)$, we can write as $v = \sum_{j=1}^\infty (v, \varphi_j)\varphi_j$, and the Parseval's relation

$$(v, w) = \sum_{j=1}^\infty (v, \varphi_j)(w, \varphi_j),$$

holds. Thus, let the space $\dot{H}^\nu = \dot{H}^\nu(\Omega)$ be defined by the norm

$$\|v\|_{\dot{H}^\nu} = \left(\sum_{j=1}^\infty \lambda_j^\nu (v, \varphi_j)^2\right)^{1/2} < \infty \quad \text{for } \nu \geq -1,\ v \in H^{-1}.$$

For $0 < \nu < 1$, we have $\dot{H}^{-\nu} = H^{-\nu}$, since both $\dot{H}^{-\nu}$ and $H^{-\nu}$ is the uniquely defined interpolation space between L^2 and H^{-1}. Notice that, $\dot{H}^{\nu} = H_0^{\nu}$ for $0 \le \nu \le 1$, and for $1 \le \nu \le 2$, $\dot{H}^{\nu}$ consists of the functions $u \in H_0^1(\Omega)$ such that Δu belongs to the negative space $H^{\nu-2}$ (cf. [9]).

Let $E(t)$ be the solution operator of problem (4.17). Then the solution $u(t)$ is represented by

$$u(t) = E(t)u_0 = \sum_{j=1}^{\infty} e^{-\lambda_j t}(u_0, \varphi_j)\varphi_j \quad \text{for } u_0 \in H^{-1},$$

where $E(t)$ is an analytic semigroup in L^2, and Δ is the generator of $E(t)$ (cf. [74, 98]). Using the Parseval's relation it follows that $E(t)$ is a contraction in L^2 and it satisfies the smoothing property

$$\|D_t^l E(t)u_0\|_{\dot{H}^{\nu_2}} \le Ct^{-(\nu_2-\nu_1)/2-l}\|u_0\|_{\dot{H}^{\nu_1}} \quad \text{for} \quad -1 \le \nu_1 \le \nu_2,\ l = 0, 1. \tag{4.19}$$

The following theorem gives the $L^\infty(L^2)$ error estimates for the semidiscrete case.

Theorem 4.3.1. *Let $u^{CFE}(t)$ and $u(t)$ be the solutions of (4.18) and (4.17), respectively and $u(t) \in H_0^1(\Omega) \cap H^{1+s}(\Omega)$, $1/2 \le s \le 1$, for each t. Then with $u_0^{CFE} = P^{CFE}u_0$ and for $1/2 \le s \le 1$, we have*

$$\|u^{CFE}(t) - u(t)\| \le CH^{2s}\widetilde{Log}^{s}(H/h)\, t^{-(1+s)/2}\|u_0\| \quad \textit{for} \ \ t > 0.$$

Proof. It follows from Lemma 3.2.2 that

$$\|u^{\text{CFE}}(t) - u(t)\| \le Ct^{-1}\sup_{\zeta\le t}(\zeta^2\|\rho_t(\zeta)\| + \zeta\|\rho(\zeta)\| + \|\hat{\rho}(\zeta)\|) \quad \text{for} \ \ t > 0, \tag{4.20}$$

where $\hat{\rho}(t) = \int_0^t \rho(\zeta)d\zeta$. Note that the smoothness of Γ is not required for proving the above inequality. Using the definition of H^{1+s} and $\dot{H}^{1+s}$ together with (4.19) and Lemma 4.2.1, we easily obtain

$$\begin{aligned} \zeta\|\rho(\zeta)\| &\le C\zeta H^{2s}\widetilde{\text{Log}}^{s}(H/h)\|u(\zeta)\|_{\dot{H}^{1+s}} \\ &\le C\zeta H^{2s}\widetilde{\text{Log}}^{s}(H/h)\zeta^{-(1+s)/2}\|u_0\| = CH^{2s}\widetilde{\text{Log}}^{s}(H/h)\zeta^{(1-s)/2}\|u_0\|. \end{aligned}$$

In a similar way,

$$\zeta^2\|\rho_t(\zeta)\| \le CH^{2s}\widetilde{\text{Log}}^{s}(H/h)\zeta^2\|u_t(\zeta)\|_{\dot{H}^{1+s}} \le CH^{2s}\widetilde{\text{Log}}^{s}(H/h)\zeta^{(1-s)/2}\|u_0\|.$$

Since $s \leq 1$, we have

$$\begin{aligned}\|\hat{\rho}(\zeta)\| &\leq \int_0^\zeta \|\rho(\tilde{\zeta})\| d\tilde{\zeta} \\ &\leq CH^{2s}\widetilde{\text{Log}}^s(H/h) \int_0^\zeta \tilde{\zeta}^{(-1-s)/2}\|u_0\| d\tilde{\zeta} \\ &\leq CH^{2s}\widetilde{\text{Log}}^s(H/h)\zeta^{(1-s)/2}\|u_0\|.\end{aligned}$$

Altogether these inequalities and (4.20) completes the rest of the proof. □

We now turn to the fully discrete backward Euler approximation of the problem (4.17). The backward Euler scheme is stated as follows: Given $U^0 = u_0^{\text{CFE}}$, seek $U^n \in S^{\text{CFE}}$ such that

$$(U^n, \chi) + k\mathcal{A}(U^n, \chi) = (U^{n-1}, \chi) \quad \forall \chi \in S^{\text{CFE}}, \ n \geq 1. \tag{4.21}$$

Next, we turn to show the error estimates for the fully discrete backward Euler method with nonsmooth initial data.

Theorem 4.3.2. *Let U^n and $u(t_n)$ be the solutions of (4.21) and (4.17), respectively. Assume that $u_0^{CFE} = P^{CFE}u_0$. Then, for $1/2 \leq s \leq 1$, we have*

$$\|U^n - u(t_n)\| \leq C(H^{2s}\widetilde{Log}^s(H/h)t_n^{-(1+s)/2} + kt_n^{-1})\|u_0\| \quad \textit{for } \ t_n > 0.$$

Proof. We split the error term as

$$\|U^n - u(t_n)\| \leq \|U^n - u^{\text{CFE}}(t_n)\| + \|u^{\text{CFE}}(t_n) - u(t_n)\|. \tag{4.22}$$

For the first term on the right-hand side of (4.22), we have from Lemma 3.2.3 that

$$\|U^n - u^{\text{CFE}}(t_n)\| \leq Ckt_n^{-1}\|P^{\text{CFE}}u_0\| \leq Ckt_n^{-1}\|u_0\| \quad \text{for } \ t_n > 0,$$

and in view of Theorem 4.3.1, we obtain the estimate for the second term. Altogether these inequalities complete the rest of the proof. □

4.4 Concluding Remarks

In this chapter, we have presented the CFE method for parabolic problem in a nonconvex polygonal domain. Both smooth and nonsmooth initial data cases are considered and analyzed. For a two-scale CFE discretization, error estimates of order $\mathcal{O}(H^{2s}\widetilde{\text{Log}}^s(H/h) + k)$ in the $L^\infty(L^2)$-norm and $\mathcal{O}(H^s\widetilde{\text{Log}}^{s/2}(H/h) + k)$ in the $L^\infty(H^1)$-norm for $1/2 \leq s \leq 1$, are proved for smooth initial data. Also, an error estimate of order $\mathcal{O}(H^{2s}\widetilde{\mathrm{Log}}^{s}(H/h)t_n^{-(1+s)/2} + kt_n^{-1})$, $1/2 \le s \le 1$, $t_n > 0$ in the $L^\infty(L^2)$-norm is established for nonsmooth data. Note that, when $s = 1$, the results of this chapter coincide with the results for the convex domains derived in Chapters 2 and 3. The present approach allows to resolve the boundary to take into account the Dirichlet boundary condition without increasing so much degrees of freedom, which is very advantageous when the domain boundary is very complicated. As the degrees of freedom for the CFE method located on the inner region Ω^{in}, the dimension of the CFE space is much less than the standard finite element space for the complicated domains. Moreover, Theorems 4.2.1, 4.2.2 and 4.2.3 for smooth initial data and Theorems 4.3.1 and 4.3.2 for nonsmooth initial data yields the same ROC as in the standard FEM for nonconvex domains when the 'two-scale CFE method' becomes 'one-scale CFE method', i.e., when the two-scale grid $\mathcal{T}_{H,h}$ coincides with the coarse-scale grid $\mathcal{T}_H$ for the case $h = \mathcal{O}(H)$. Numerical experiment for both smooth and nonsmooth initial data are presented in Chapter 7 (see Examples 7.3 and 7.4) to support our theoretical analysis.

5

Two-Scale CFE Method for Nonlinear Parabolic Problems in Nonconvex Polygonal Domains

In the previous chapters, we have considered a two-scale CFEs for the approximation of linear parabolic IBVPs with the Dirichlet boundary conditions. In this chapter, we discuss *a priori* error analysis for the spatially semidiscrete and fully discrete CFE approximations of nonlinear parabolic IBVP of the form (1.3) with the Dirichlet boundary condition in a nonconvex polygonal domain. A continuous, piecewise linear CFE space is employed for the spatially semidiscrete finite element approximation and the temporal discretization is based on the backward Euler scheme at equidistant time step. The modified linearized backward Euler scheme is also considered for the temporal discretizations. We derive almost optimal order convergence (up to logarithmic terms) in space and optimal order in time for the CFE method in the $L^\infty(L^2)$-norm.

5.1 Introduction

To start with, we first recall the nonlinear parabolic IBVP:

$$
\begin{aligned}
u_t - \nabla\cdot(a(u)\nabla u) &= f(u) && \text{in } \Omega_T,\\
u &= 0 && \text{on } \Gamma_T,\\
\text{with } u(\cdot,0) &= u_0 && \text{in } \Omega,
\end{aligned}
\tag{5.1}
$$

where $u = u(x,t)$, $\Omega_T := \Omega\times J$ and $\Gamma_T := \Gamma\times J$ with $J := (0,T]$, $T < \infty$. The spatial domain Ω is assumed be a nonconvex polygonal domain in $\mathbb{R}^2$ with boundary Γ. Further, we assume that the functions a and f are smooth on $\mathbb{R}$ such that

$$
0 < \widehat{\mu} \le a(u) \le M, \quad |a'(u)| + |f'(u)| \le \widehat{B} \quad \text{for } u\in\mathbb{R},
\tag{5.2}
$$

where $\widehat{\mu}$, M and $\widehat{B}$ are some positive constants and the above problem admits a unique solution.

The weak formulation of the problem (5.1) is written as: Find $u : \bar{J} \to H_0^1(\Omega)$ such that

$$(u_t, \varphi) + (a(u)\nabla u, \nabla \varphi) = (f(u), \varphi) \quad \forall \varphi \in H_0^1(\Omega),\ t \in J,$$
$$u(0) = u_0(x).$$

In order to pose the semidiscrete problem we recall the CFE space S^{CFE}. We first define the continuous and piecewise linear finite element space S corresponding to the triangulation $\mathcal{T}_{H,h}$ as

$$S := \{v \in C^0(\Omega_{H,h}) \ | \ v|_K \in \mathbb{P}_1 \ \ \forall K \in \mathcal{T}_{H,h}\},$$

where $\Omega_{H,h} := \mathrm{int}\left(\bigcup_{K \in \mathcal{T}_{H,h}} K\right)$. Then using the definition of the extrapolation operator $\mathcal{E} : \mathbb{R}^{\vartheta_{\mathrm{dof}}} \to \mathbb{R}^{\vartheta_{H,h}}$ defined in (1.11), we now write the CFE space S^{CFE} corresponding to the triangulation $\mathcal{T}_{H,h}$

$$S^{\mathrm{CFE}} := \{v \in S \mid \exists\, \mathbf{\Phi} \in \mathbb{R}^{\vartheta_{\mathrm{dof}}} \ \text{ with } v(x) = (\mathcal{E}\mathbf{\Phi})_x \ \ \forall x \in \vartheta_{H,h}\},$$

where $\mathbf{\Phi} : \vartheta_{\mathrm{dof}} \to \mathbb{R}$ is the grid function. Note that, S^{CFE} is a subspace of S, where the values at the slave nodes are restricted by the extrapolation. For a domain with very complicated boundary Γ, the dimension of S^{CFE} is very less than the dimension of S because the number of nodes in ϑ_{dof} determines the dimension of S^{CFE}. We now pose the semidiscrete problem as to find $u^{\mathrm{CFE}} : \bar{J} \to S^{\mathrm{CFE}}$ such that

$$(u_t^{\mathrm{CFE}}, \chi) + (a(u^{\mathrm{CFE}})\nabla u^{\mathrm{CFE}}, \nabla \chi) = (f(u^{\mathrm{CFE}}), \chi) \quad \forall \chi \in S^{\mathrm{CFE}},\ t \in J, \tag{5.3}$$
$$u^{\mathrm{CFE}}(0) = u_0^{\mathrm{CFE}},$$

where u_0^{CFE} is an approximation of u_0 in S^{CFE}.

For the fully discretization scheme we use the backward Euler scheme. For the constant time step k with $t_n = nk$, consider the backward Euler quotient $\bar{\partial} U^n = (U^n - U^{n-1})/k$, where U^n is the approximation of $u(t_n)$ in S^{CFE}. Then the fully discrete backward Euler approximation of the problem (5.1) associated with the CFE space S^{CFE} is stated as follows: Given $U^0 = u_0^{\mathrm{CFE}}$, seek $U^n \in S^{\mathrm{CFE}}$ for $n \geq 1$ such that

$$(\bar{\partial} U^n, \chi) + (a(U^n)\nabla U^n, \nabla \chi) = (f(U^n), \chi) \quad \forall \chi \in S^{\mathrm{CFE}},\ t_n \in J. \tag{5.4}$$

We also consider the modified linearized backward Euler scheme. By replacing the terms $a(U^n)$ and $f(U^n)$ in (5.4) by $a(U^{n-1})$ and $f(U^{n-1})$, respectively. Then the linearized modification for the backward Euler scheme is stated as follows: Given $U^0 = u_0^{\text{CFE}}$, seek $U^n \in S^{\text{CFE}}$ such that

$$(\bar{\partial}U^n, \chi) + (a(U^{n-1})\nabla U^n, \nabla\chi) = (f(U^{n-1}), \chi) \quad \forall \chi \in S^{\text{CFE}},\ t_n \in J.$$

A priori error analysis for the Galerkin finite element approximation of nonlinear parabolic problems has been investigated by numerous authors, see Dendy [31], Douglas [32], Douglas and Dupont [35], Gao *et al.* [43], Luskin [68], Rachford [82], Li and Zhang [64] and Thomée [102]. In [68], Luskin has analyzed the Galerkin method for nonlinear parabolic equations with nonlinear boundary conditions, where an extrapolation in time is used to yield a system of linear algebraic equations to be solved at each time level. Thomée [102] has developed the standard Galerkin FEM for the approximation of nonlinear parabolic problem in convex domains, where the discretization with respect to space is done by piecewise linear finite elements and in time the backward Euler and Crank-Nicolson methods are applied. The author has proved an optimal order convergence $\mathcal{O}(h^2)$ with respect to space in the $L^\infty(L^2)$-norm for the spatially semidiscrete scheme. With respect to time an optimal order convergence $\mathcal{O}(k)$ and $\mathcal{O}(k^2)$ for the backward Euler and the Crank-Nicolson method, respectively has been proved. The stability and convergence of fully discrete Galerkin FEM for the nonlinear thermistor equations in a nonconvex polygonal domain has been investigated by Gao *et al.* [43]. They have established the unconditional stability and optimal error estimates of a linearized backward Euler Galerkin FEM for the time-dependent nonlinear thermistor equations in a two-dimensional nonconvex polygonal domain.

In this chapter, we have considered a two-scale CFEs for the approximation of nonlinear parabolic IBVP with the Dirichlet boundary condition in a nonconvex polygonal domain. The key idea in the error analysis is the error splitting technique and the estimates for the associated elliptic or Ritz projection in the framework of CFE method. We derive *a priori* error bounds in the $L^\infty(L^2)$-norm for both the semidiscrete and fully discrete CFE method.

The outline of this chapter is as follows. In Section 5.2, we have presented some auxiliary results which will be needed in the error analysis. The spatially semidiscrete and fully discrete CFE error analysis are presented in this section. Some concluding remarks are presented in the last section.

5.2 CFE Error Estimates

Before we start the error analysis, we first show the existence and uniqueness of the solution of (5.3). Following the similar argument as described in Chapter 2, let

$\{\phi_i^{\mathrm{CFE}}\}_{i=1}^{N^{\mathrm{CFE}}}$ be the nodal basis functions for the CFE space S^{CFE} defined by

$$\phi_i^{\mathrm{CFE}} := \mathcal{E}(\varrho_i) \in S^{\mathrm{CFE}} \quad \forall\, 1 \le i \le N^{\mathrm{CFE}},$$

where $\{\varrho_i\}_{i=1}^{N^{\mathrm{CFE}}}$ be the standard nodal basis function of the finite element space S^{in} as defined in (2.7). For each free node $x_i \in \vartheta_{\mathrm{dof}}$, we can associate a corresponding basis function ϕ_i^{CFE} as in the case of standard finite elements. Thus, we have

$$\phi_i^{\mathrm{CFE}}(x_j) = \delta_{ij} \quad \forall\, 1 \le i, j \le N^{\mathrm{CFE}}.$$

Representing the solution in terms of the basis functions $\{\phi_j^{\mathrm{CFE}}\}_{j=1}^{N^{\mathrm{CFE}}}$ as $u^{\mathrm{CFE}}(x,t) = \sum_{j=1}^{N^{\mathrm{CFE}}} \alpha_j(t)\phi_j^{\mathrm{CFE}}(x)$, (5.3) may be written as

$$\sum_{j=1}^{N^{\mathrm{CFE}}} \alpha_j'(t)(\phi_j^{\mathrm{CFE}}, \phi_k^{\mathrm{CFE}}) + \sum_{j=1}^{N^{\mathrm{CFE}}} \alpha_j(t)\big(a(\sum_{l=1}^{N^{\mathrm{CFE}}} \alpha_l(t)\phi_l^{\mathrm{CFE}})\nabla\phi_j^{\mathrm{CFE}}, \nabla\phi_k^{\mathrm{CFE}}\big)$$
$$= \big(f(\sum_{l=1}^{N^{\mathrm{CFE}}} \alpha_l(t)\phi_l^{\mathrm{CFE}}), \phi_k^{\mathrm{CFE}}\big), \quad k = 1, 2, ..., N^{\mathrm{CFE}}.$$

This can be written in the matrix form by setting $\alpha = \alpha(t) = (\alpha_1(t), \dots, \alpha_{N^{\mathrm{CFE}}}(t))^T$ as

$$A^{\mathrm{CFE}}\,\alpha' + B^{\mathrm{CFE}}(\alpha)\,\alpha = f^{\mathrm{CFE}}(\alpha) \quad \text{for } \ t \in J, \ \text{ with } \ \alpha(0) = \gamma,$$

where γ is the vector of the nodal values of u_0^{CFE}. Here the matrices $A^{\mathrm{CFE}} = \big(A_{jk}^{\mathrm{CFE}}\big)$ and $B^{\mathrm{CFE}}(\alpha) = \big(B_{jk}^{\mathrm{CFE}}(\alpha)\big)$ with elements

$$A_{jk}^{\mathrm{CFE}} = (\phi_j^{\mathrm{CFE}}, \phi_k^{\mathrm{CFE}}) \quad \text{and} \quad B_{jk}^{\mathrm{CFE}}(\alpha) = \big(a(\sum_{l=1}^{N^{\mathrm{CFE}}} \alpha_l\phi_l^{\mathrm{CFE}})\nabla\phi_j^{\mathrm{CFE}}, \nabla\phi_k^{\mathrm{CFE}}\big),$$

respectively, and $f^{\mathrm{CFE}}(\alpha) = \big(f_j^{\mathrm{CFE}}(\alpha)\big)$ is the vector with entries

$$f_j^{\mathrm{CFE}}(\alpha) = \big(f(\sum_{l=1}^{N^{\mathrm{CFE}}} \alpha_l\phi_l^{\mathrm{CFE}}), \phi_j^{\mathrm{CFE}}\big).$$

Using our assumption (5.2), it can be easily derive that the matrices A^{CFE} and $B^{\mathrm{CFE}}(\alpha)$ are positive definite, and hence in particular, A^{CFE} is invertible. Also, $B^{\mathrm{CFE}}(\alpha)$ and $f^{\mathrm{CFE}}(\alpha)$ are globally Lipschitz continuous on $\mathbb{R}^{N^{\mathrm{CFE}}}$. It follows easily that the system has a unique solution which is bounded for $t \in J$.

Before we start the error estimates for the semidiscrete problem (5.3), we need the following auxiliary result. For a proof, see Lemma 4.2.1.

Lemma 5.2.1. *Let $w \in H_0^1(\Omega) \cap H^{1+s}(\Omega)$ for $1/2 \leq s \leq 1$ and $R^{CFE} : H_0^1(\Omega) \to S^{CFE}$ be defined by (4.4). Also let the conditions (1.9) and (2.9) hold true. Then*

$$\|\nabla(w - R^{CFE}w)\| \leq CH^s \widetilde{Log}^{s/2}(H/h)\|w\|_{1+s,\Omega} \quad \textit{for } 1/2 \leq s \leq 1,$$

where the constant C depends on σ_{dist}, σ_e, σ_{uni}, $C_{ol,1}$, $C_{ol,2}$, σ_{ext} and the minimal angles in the triangulation $\mathcal{T}_{H,h}$ (as described in Chapter 2).

5.2.1 Spatially Semidiscrete Error Estimates

This section concerns about the error analysis for the spatially semidiscrete problem (5.3). To begin with, we define the elliptic projection $\tilde{u}^{\mathrm{CFE}} = \tilde{u}^{\mathrm{CFE}}(t)$ of the exact solution u of (5.1) in S^{CFE} as

$$\big(a(u(t))\nabla(\tilde{u}^{\mathrm{CFE}}(t) - u(t)), \nabla\chi\big) = 0 \quad \forall \chi \in S^{\mathrm{CFE}},\ t \geq 0. \tag{5.5}$$

In order to estimate the error for the semidiscrete problem (5.3), we split the error term as follows:

$$u^{\mathrm{CFE}} - u = (u^{\mathrm{CFE}} - \tilde{u}^{\mathrm{CFE}}) + (\tilde{u}^{\mathrm{CFE}} - u) = \tilde{\theta} + \tilde{\rho}. \tag{5.6}$$

The next result shows the estimate for $\tilde{\rho}$ in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms.

Lemma 5.2.2. *Let $\tilde{\rho}$ be defined by (5.6), and let (5.2) hold true. Then, under the appropriate regularity assumptions on u, we have for $1/2 \leq s \leq 1$,*

$$\|\nabla\tilde{\rho}(t)\| \leq C(u)H^s\widetilde{Log}^{s/2}(H/h), \quad \textit{and} \quad \|\tilde{\rho}(t)\| \leq C(u)H^{2s}\widetilde{Log}^{s}(H/h),$$

where $C(u)$ is independent of $t \in J$.

Proof. In view of (5.5) and for any $\chi \in S^{\mathrm{CFE}}$, we have

$$\begin{aligned}
\widehat{\mu}\|\nabla(\tilde{u}^{\mathrm{CFE}} - u)\|^2 &\leq (a(u)\nabla(\tilde{u}^{\mathrm{CFE}} - u), \nabla(\tilde{u}^{\mathrm{CFE}} - u)) \\
&= (a(u)\nabla(\tilde{u}^{\mathrm{CFE}} - u), \nabla(\chi - u)) \\
&\leq M\|\nabla(\tilde{u}^{\mathrm{CFE}} - u)\|\|\nabla(\chi - u)\|,
\end{aligned}$$

and hence,

$$\|\nabla(\tilde{u}^{\mathrm{CFE}} - u)\| \leq (M/\widehat{\mu})\|\nabla(\chi - u)\|.$$

Choosing $\chi = R^{\mathrm{CFE}}u$ and using Lemma 5.2.1, we obtain

$$\|\nabla(\tilde{u}^{\mathrm{CFE}} - u)\| \leq CH^s \widetilde{\mathrm{Log}}^{s/2}(H/h)\|u\|_{1+s,\Omega},$$

which proves the estimate for $\|\nabla\tilde{\rho}(t)\|$. To estimate $\|\tilde{\rho}(t)\|$, we use the duality argument. For $\varphi \in L^2(\Omega)$, consider the problem:

$$-\nabla \cdot (a(u)\nabla\psi) \equiv -a(u)\Delta\psi - \nabla a(u)\cdot\nabla\psi = \varphi \quad \text{in } \Omega, \qquad \text{with} \quad \psi = 0 \quad \text{on } \Gamma. \tag{5.7}$$

Since $\|\psi\| \leq C\|\nabla\psi\|$ for $\psi \in H_0^1(\Omega)$, we have

$$\widehat{\mu}\|\nabla\psi\|^2 \leq (a(u)\nabla\psi, \nabla\psi) = (\varphi, \psi) \leq \|\varphi\|\|\psi\| \leq C\|\varphi\|\|\nabla\psi\|,$$

which gives $\|\nabla\psi\| \leq C\|\varphi\|$. Since $\nabla a(u) = a'(u)\nabla u$, using the regularity estimate (see [8, 22]) and with the help of (5.2), we have

$$\|\psi\|_{1+s} \leq C\|\Delta\psi\| \leq C\|a(u)\Delta\psi\| = C\|\varphi + \nabla a(u)\cdot\nabla\psi\| \leq C\|\varphi\|, \tag{5.8}$$

for $1/2 \leq s \leq 1$. Setting $\chi = R^{\mathrm{CFE}}\psi$ together with (5.8), we obtain

$$\begin{aligned}
(\tilde{u}^{\mathrm{CFE}} - u, \varphi) &= (a(u)\nabla(\tilde{u}^{\mathrm{CFE}} - u), \nabla\psi) \\
&= (a(u)\nabla(\tilde{u}^{\mathrm{CFE}} - u), \nabla(\psi - \chi)) \\
&\leq M\|\nabla(\tilde{u}^{\mathrm{CFE}} - u)\|\|\nabla(\psi - \chi)\| \\
&\leq (CH^s\widetilde{\mathrm{Log}}^{s/2}(H/h)\|u\|_{1+s})(CH^s\widetilde{\mathrm{Log}}^{s/2}(H/h)\|\psi\|_{1+s}) \\
&\leq CH^{2s}\widetilde{\mathrm{Log}}^{s}(H/h)\|u\|_{1+s}\|\varphi\|,
\end{aligned}$$

and hence

$$\|\tilde{u}^{\mathrm{CFE}} - u\| \leq CH^{2s}\widetilde{\mathrm{Log}}^{s}(H/h)\|u\|_{1+s,\Omega}.$$

This completes the proof. □

Next, we concentrate on the estimates of $\tilde{\rho}_t$ in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms.

Lemma 5.2.3. *Let $\tilde{\rho}$ be defined by (5.6), and let (5.2) hold true. Then, under the appropriate regularity assumptions on u, we have for $1/2 \leq s \leq 1$,*

$$\|\nabla\tilde{\rho}_t(t)\| \leq C(u)H^s\widetilde{Log}^{s/2}(H/h), \quad \text{and} \quad \|\tilde{\rho}_t(t)\| \leq C(u)H^{2s}\widetilde{Log}^{s}(H/h),$$

where $C(u)$ is independent of $t \in J$.

Proof. To begin, we differentiate (5.5) to obtain

$$(a(u)\nabla\tilde{\rho}_t, \nabla\chi) + (a(u)_t\nabla\tilde{\rho}, \nabla\chi) = 0 \quad \forall\chi \in S^{\mathrm{CFE}}. \tag{5.9}$$

By the uniformly boundedness of $a(u)$ and $a(u)_t$ together with using (5.9), we have

$$\begin{aligned}
\widehat{\mu}\|\nabla\tilde{\rho}_t\|^2 &\leq (a(u)\nabla\tilde{\rho}_t, \nabla\tilde{\rho}_t)\\
&= (a(u)\nabla\tilde{\rho}_t, \nabla(\chi - u_t)) + (a(u)\nabla\tilde{\rho}_t, \nabla(\tilde{u}_t^{\mathrm{CFE}} - \chi))\\
&= (a(u)\nabla\tilde{\rho}_t, \nabla(\chi - u_t)) + (a(u)_t\nabla\tilde{\rho}, \nabla(\chi - \tilde{u}_t^{\mathrm{CFE}}))\\
&\leq C(\|\nabla\tilde{\rho}_t\|\|\nabla(\chi - u_t)\| + \|\nabla\tilde{\rho}\|\|\nabla(\chi - \tilde{u}_t^{\mathrm{CFE}})\|)\\
&\leq C\big(\|\nabla\tilde{\rho}_t\|\|\nabla(\chi - u_t)\| + \|\nabla\tilde{\rho}\|(\|\nabla(\chi - u_t)\| + \|\nabla(u_t - \tilde{u}_t^{\mathrm{CFE}})\|)\big).
\end{aligned}$$

Setting $\chi = R^{\mathrm{CFE}}u_t$, we have

$$\widehat{\mu}\|\nabla\tilde{\rho}_t\|^2 \leq \frac{\widehat{\mu}}{2}\|\nabla\tilde{\rho}_t\|^2 + C\|\nabla\tilde{\rho}\|^2 + C(u)H^{2s}\widetilde{\mathrm{Log}}^{s}(H/h),$$

and using the previous estimate of $\nabla\tilde{\rho}$ already shown, we have

$$\|\nabla\tilde{\rho}_t\| \leq C(u)H^s\widetilde{\mathrm{Log}}^{s/2}(H/h). \tag{5.10}$$

Now, in order to estimate $\|\tilde{\rho}_t\|$, we again use the duality argument as in the proof of Lemma 5.2.2. Let ψ be defined as in (5.7), then using (5.9) we write

$$\begin{aligned}
(\tilde{\rho}_t, \varphi) &= (a(u)\nabla\tilde{\rho}_t, \nabla\psi)\\
&= (a(u)\nabla\tilde{\rho}_t, \nabla(\psi - \chi)) + (a(u)\nabla\tilde{\rho}_t, \nabla\chi)\\
&= (a(u)\nabla\tilde{\rho}_t, \nabla(\psi - \chi)) - (a(u)_t\nabla\tilde{\rho}, \nabla\chi)\\
&= (a(u)\nabla\tilde{\rho}_t, \nabla(\psi - \chi)) + (a(u)_t\nabla\tilde{\rho}, \nabla(\psi - \chi)) - (\nabla\tilde{\rho}, a(u)_t\nabla\psi).
\end{aligned}$$

Choose $\chi = R^{\mathrm{CFE}}\psi$ in the above and use integration by parts formula for the last term, we obtain

$$\begin{aligned}
|(\tilde{\rho}_t, \varphi)| \leq C(&\|\nabla\tilde{\rho}_t\|H^s\widetilde{\mathrm{Log}}^{s/2}(H/h)\|\psi\|_{1+s} + \|\nabla\tilde{\rho}\|H^s\widetilde{\mathrm{Log}}^{s/2}(H/h)\|\psi\|_{1+s}\\
&+ \|\tilde{\rho}\|\|\nabla\cdot(a(u)_t\nabla\psi)\|).
\end{aligned} \tag{5.11}$$

In view of (5.8) and for $\nabla a(u)_t$ bounded, the last term gives

$$\begin{aligned}
\|\nabla\cdot(a(u)_t\nabla\psi)\| &= \|\nabla a(u)_t\cdot\nabla\psi + a(u)_t\Delta\psi\|\\
&\leq C(\|\nabla\psi\| + \|a(u)_t\Delta\psi\|)\\
&\leq C(\|\varphi\| + \|\Delta\psi\|) \leq C\|\varphi\|,
\end{aligned}$$

which together with Lemma 5.2.2, equations (5.10) and (5.11) yields

$$|(\tilde{\rho}_t, \varphi)| \leq C(u)H^{2s}\widetilde{\mathrm{Log}}^s(H/h)\|\varphi\|,$$

and hence, $\|\tilde{\rho}_t\| \leq C(u)H^{2s}\widetilde{\mathrm{Log}}^s(H/h)$. This completes the proof of the lemma. □

We now derive the main result of this section. The following theorem shows an estimate for the error in the $L^\infty(L^2)$-norm for the semidiscrete problem (5.3).

Theorem 5.2.1. *Let u^{CFE} and u be the solutions of (5.3) and (5.1), respectively. Then, under the assumption of (5.2), we have*

$$\|u^{CFE}(t) - u(t)\| \leq C\|u_0^{CFE} - u_0\| + C(u)H^{2s}\widetilde{Log}^s(H/h)$$

for $1/2 \leq s \leq 1$, $t \in \bar{J}$.

Proof. To begin, we split the error as in (5.6). In view of Lemma 5.2.2, it is enough to estimate $\tilde{\theta}$. Using (5.5), we have for $\chi \in S^{\mathrm{CFE}}$,

$$\begin{aligned}
&(\tilde{\theta}_t, \chi) + (a(u^{\mathrm{CFE}})\nabla\tilde{\theta}, \nabla\chi) \\
&\quad = (u_t^{\mathrm{CFE}}, \chi) + (a(u^{\mathrm{CFE}})\nabla u^{\mathrm{CFE}}, \nabla\chi) - (\tilde{u}_t^{\mathrm{CFE}}, \chi) - (a(u^{\mathrm{CFE}})\nabla\tilde{u}^{\mathrm{CFE}}, \nabla\chi) \\
&\quad = (f(u^{\mathrm{CFE}}), \chi) - (\tilde{u}_t^{\mathrm{CFE}} - u_t, \chi) - (u_t, \chi) - (a(u)\nabla\tilde{u}^{\mathrm{CFE}}, \nabla\chi) \\
&\qquad\qquad + ((a(u) - a(u^{\mathrm{CFE}}))\nabla\tilde{u}^{\mathrm{CFE}}, \nabla\chi) \\
&\quad = (f(u^{\mathrm{CFE}}), \chi) - (\tilde{\rho}_t, \chi) - (u_t, \chi) - (a(u)\nabla u, \nabla\chi) \\
&\qquad\qquad + ((a(u) - a(u^{\mathrm{CFE}}))\nabla\tilde{u}^{\mathrm{CFE}}, \nabla\chi) \\
&\quad = (f(u^{\mathrm{CFE}}) - f(u), \chi) + ((a(u) - a(u^{\mathrm{CFE}}))\nabla\tilde{u}^{\mathrm{CFE}}, \nabla\chi) - (\tilde{\rho}_t, \chi).
\end{aligned} \tag{5.12}$$

Now, using (5.2) and (5.5), we have

$$\begin{aligned}
\widehat{\mu}\|\nabla\tilde{u}^{\mathrm{CFE}}\|^2 &= \widehat{\mu}(\nabla\tilde{u}^{\mathrm{CFE}}, \nabla\tilde{u}^{\mathrm{CFE}}) \\
&\leq (a(u)\nabla\tilde{u}^{\mathrm{CFE}}, \nabla\tilde{u}^{\mathrm{CFE}}) \\
&= (a(u)\nabla u, \nabla\tilde{u}^{\mathrm{CFE}}) \leq M(\nabla u, \nabla\tilde{u}^{\mathrm{CFE}}),
\end{aligned}$$

which leads to

$$\|\nabla\tilde{u}^{\mathrm{CFE}}\| \leq (M/\widehat{\mu})\|\nabla u\| = C\|\nabla u\|. \tag{5.13}$$

Now choosing $\chi = \tilde{\theta}$ in (5.12), then using (5.2) and (5.13), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\tilde{\theta}\|^2 + \widehat{\mu}\|\nabla\tilde{\theta}\|^2 &\leq C\big(\|u^{\mathrm{CFE}} - u\|(\|\tilde{\theta}\| + \|\nabla\tilde{\theta}\|) + \|\tilde{\rho}_t\|\|\tilde{\theta}\|\big) \\
&\leq \widehat{\mu}\|\nabla\tilde{\theta}\|^2 + C(\|\tilde{\theta}\|^2 + \|\tilde{\rho}\|^2 + \|\tilde{\rho}_t\|^2).
\end{aligned}$$

An integration from 0 to t leads to

$$\|\tilde{\theta}(t)\|^2 \leq \left(\|\tilde{\theta}(0)\|^2 + C\int_0^t (\|\tilde{\rho}\|^2 + \|\tilde{\rho}_t\|^2)d\zeta\right) + C\int_0^t \|\tilde{\theta}\|^2 d\zeta.$$

We apply Gronwall's lemma to obtain

$$\|\tilde{\theta}(t)\|^2 \leq C\|\tilde{\theta}(0)\|^2 + C\int_0^t (\|\tilde{\rho}\|^2 + \|\tilde{\rho}_t\|^2)d\zeta, \tag{5.14}$$

where C is now depending on T. Note that,

$$\begin{aligned} \|\tilde{\theta}(0)\| &\leq \|u_0^{\text{CFE}} - u_0\| + \|\tilde{u}^{\text{CFE}}(0) - u_0\| \\ &\leq \|u_0^{\text{CFE}} - u_0\| + CH^{2s}\widetilde{\text{Log}}^s(H/h)\|u_0\|_{1+s}. \end{aligned} \tag{5.15}$$

Now, altogether (5.14), (5.15), Lemmas 5.2.2 and 5.2.3 yields

$$\|\tilde{\theta}(t)\| \leq C\|u_0^{\text{CFE}} - u_0\| + C(u)H^{2s}\widetilde{\text{Log}}^s(H/h),$$

and this completes the proof. □

5.2.2 Fully Discrete Error Estimates

This section considers the discretization with respect to time variable of the spatially semidiscrete problem (5.3). We begin with the *backward Euler* CFE method and derive some *a priori* error estimates. Let k be the time step with $t_n = nk$ and consider $\bar{\partial}U^n = (U^n - U^{n-1})/k$, where U^n is the approximation of $u(t_n)$ in S^{CFE}. The backward Euler CFE method, for $n \geq 1$, is given by

$$\begin{aligned} (\bar{\partial}U^n, \chi) + (a(U^n)\nabla U^n, \nabla\chi) &= (f(U^n), \chi) \quad \forall \chi \in S^{\text{CFE}},\ t_n \in J, \\ U^0 &= u_0^{\text{CFE}}. \end{aligned} \tag{5.16}$$

We write $U^n = \sum_{j=1}^{N^{\text{CFE}}} \alpha_j^n \phi_j^{\text{CFE}}$, then the matrix form of the equation (5.16) can be written as

$$A^{\text{CFE}}\frac{\alpha^n - \alpha^{n-1}}{k} + B^{\text{CFE}}(\alpha^n)\,\alpha^n = f^{\text{CFE}}(\alpha^n),$$

where α^n is the unknown vector. Which after simplification leads to

$$(A^{\text{CFE}} + kB^{\text{CFE}}(\alpha^n))\alpha^n = A^{\text{CFE}}\alpha^{n-1} + kf^{\text{CFE}}(\alpha^n) \quad \text{for } t_n \in J,$$

$$\text{with} \quad \alpha^0 = \gamma,$$

where A^{CFE}, $B^{\text{CFE}}(\alpha)$, $f^{\text{CFE}}(\alpha)$ and γ are defined as before. Now we show the existence and uniqueness for the solution of equation (5.16).

Existence: In order to show the existence, we proceed as in [102]. Multiply (5.16) by $2k$ and write it as $(F^{\text{CFE}}(U^n), \chi) = 0 \ \forall \chi \in S^{\text{CFE}}$. Clearly, $F^{\text{CFE}} : S^{\text{CFE}} \to S^{\text{CFE}}$ is a continuous function. Now, for $\|\chi\|$ is large enough and with the condition (5.2), we have

$$(F^{\text{CFE}}(\chi), \chi) = 2(\chi - U^{n-1}, \chi) + 2k(a(\chi)\nabla\chi, \nabla\chi) - 2k(f(\chi), \chi)$$
$$\geq \|\chi\|^2 - \|U^{n-1}\|^2 - Ck(1 + \|\chi\|)\|\chi\| > 0,$$

provided $k \leq k' < 1/C$. It is known from the of Brouwer's fixed point theorem that if $(F^{\text{CFE}}(\chi), \chi) > 0$ for $\|\chi\| = \tilde{p}$, then the equation $F^{\text{CFE}}(Y) = 0$ has a solution $Y \in B_{\tilde{p}} = \{\chi \in S^{\text{CFE}} : \|\chi\| \leq \tilde{p}\}$. The proof is by contradiction. Let us assume that $F^{\text{CFE}}(\chi) \neq 0$ in $B_{\tilde{p}}$. Then we can easily construct a continuous map $\Psi^{\text{CFE}} : B_{\tilde{p}} \to B_{\tilde{p}}$ such that $\Psi^{\text{CFE}}(\chi) = -\tilde{p}F^{\text{CFE}}(\chi)/\|F^{\text{CFE}}(\chi)\|$. Therefore it has a fixed point $\bar{\chi}$ in $B_{\tilde{p}}$ such that

$$\Psi^{\text{CFE}}(\bar{\chi}) = \bar{\chi},$$

this imply

$$\tilde{p}^2 = \|\bar{\chi}\|^2 = -\tilde{p}(F^{\text{CFE}}(\bar{\chi}), \bar{\chi})/\|F^{\text{CFE}}(\bar{\chi})\|,$$

which contradicts the fact $(F^{\text{CFE}}(\bar{\chi}), \bar{\chi}) > 0$. This proves the existence of the solution of (5.16).

To prove the uniqueness of solution, we first show the following error estimate which is valid for any solution of (5.16).

Theorem 5.2.2. *Let U^n and u be the solutions of (5.16) and (5.1), respectively. Then under the assumption (5.2) and for small k, we have*

$$\|U^n - u(t_n)\| \leq C\|u_0^{CFE} - u_0\| + C(u)(H^{2s}\widetilde{Log}^s(H/h) + k),$$

for $t_n \in \bar{J}$, $1/2 \leq s \leq 1$.

Proof. We borrow the proof technique of [102]. We first decompose the error in two parts $\tilde{\theta}$ and $\tilde{\rho}$ as follows:

$$U^n - u^n = (U^n - \tilde{U}^n) + (\tilde{U}^n - u^n) = \tilde{\theta}^n + \tilde{\rho}^n, \tag{5.17}$$

where $u^n = u(t_n)$ and $\tilde{U}^n = \tilde{u}^{\mathrm{CFE}}(t_n)$ with $\tilde{u}^{\mathrm{CFE}}(t_n)$ be the elliptic projection of u^n defined in (5.5). Since $\tilde{\rho}^n$ is bounded by Lemma 5.2.2, it only remains to estimate $\tilde{\theta}^n$. Using the elliptic projection defined in (5.5) and for $\chi \in S^{\mathrm{CFE}}$, we note that

$$\begin{aligned}
&(\bar{\partial}\tilde{\theta}^n, \chi) + (a(U^n)\nabla\tilde{\theta}^n, \nabla\chi)\\
&\qquad = (\bar{\partial}U^n, \chi) - (\bar{\partial}\tilde{U}^n, \chi) + (a(U^n)\nabla U^n, \nabla\chi) - (a(U^n)\nabla\tilde{U}^n, \nabla\chi)\\
&\qquad = (f(U^n), \chi) - (u_t^n, \chi) - (\bar{\partial}\tilde{U}^n - u_t^n, \chi) - (a(u^n)\nabla\tilde{U}^n, \nabla\chi)\\
&\qquad\qquad - ((a(U^n) - a(u^n))\nabla\tilde{U}^n, \nabla\chi)\\
&\qquad = (f(U^n), \chi) - (u_t^n, \chi) - (\bar{\partial}\tilde{U}^n - u_t^n, \chi) - (a(u^n)\nabla u^n, \nabla\chi)\\
&\qquad\qquad - ((a(U^n) - a(u^n))\nabla\tilde{U}^n, \nabla\chi)\\
&\qquad = (f(U^n) - f(u^n), \chi) - (\bar{\partial}\tilde{\rho}^n, \chi) - (\bar{\partial}u^n - u_t^n, \chi)\\
&\qquad\qquad - ((a(U^n) - a(u^n))\nabla\tilde{U}^n, \nabla\chi),
\end{aligned}$$

and hence,

$$\begin{aligned}
(\bar{\partial}\tilde{\theta}^n, \chi) + (a(U^n)\nabla\tilde{\theta}^n, \nabla\chi) &= (f(U^n) - f(u^n), \chi) - (\bar{\partial}\tilde{\rho}^n, \chi) - (\bar{\partial}u^n - u_t^n, \chi)\\
&\qquad - ((a(U^n) - a(u^n))\nabla\tilde{U}^n, \nabla\chi).
\end{aligned}$$

With $\chi = \tilde{\theta}^n$, we use (5.2) and the boundedness of $\nabla\tilde{U}^n$ as in (5.13) to obtain

$$\frac{1}{2}\bar{\partial}\|\tilde{\theta}^n\|^2 + \widehat{\mu}\|\nabla\tilde{\theta}^n\|^2 \le C\|U^n - u^n\|(\|\tilde{\theta}^n\| + \|\nabla\tilde{\theta}^n\|) + (\|\bar{\partial}\tilde{\rho}^n\| + \|\bar{\partial}u^n - u_t^n\|)\|\tilde{\theta}^n\|.$$

Using (5.17) and after simplification we obtain

$$\begin{aligned}
\bar{\partial}\|\tilde{\theta}^n\|^2 + \widehat{\mu}\|\nabla\tilde{\theta}^n\|^2 &\le C(\|\tilde{\theta}^n\|^2 + \|\bar{\partial}\tilde{\rho}^n\|^2 + \|\tilde{\rho}^n\|^2 + \|\bar{\partial}u^n - u_t^n\|^2)\\
&= C(\|\tilde{\theta}^n\|^2 + Q_n), \qquad (5.18)
\end{aligned}$$

where $Q_n = \|\bar{\partial}\tilde{\rho}^n\|^2 + \|\tilde{\rho}^n\|^2 + \|\bar{\partial}u^n - u_t^n\|^2$. Multiply both sides of (5.18) by k to obtain

$$(1 - Ck)\|\tilde{\theta}^n\|^2 \le \|\tilde{\theta}^{n-1}\|^2 + CkQ_n,$$

which, for small k, yields

$$\|\tilde{\theta}^n\|^2 \le (1 + Ck)\|\tilde{\theta}^{n-1}\|^2 + CkQ_n.$$

A repeated application leads to

$$\|\tilde{\theta}^n\|^2 \leq (1+Ck)^n\|\tilde{\theta}^0\|^2 + Ck\sum_{j=1}^{n}(1+Ck)^{n-j}Q_j$$

$$\leq C\|\tilde{\theta}(0)\|^2 + Ck\sum_{j=1}^{n}Q_j. \tag{5.19}$$

Now, we estimate Q_j which includes three different terms. Following [102], together with Lemmas 5.2.2 and 5.2.3 we obtain

$$\|\tilde{\rho}^j\| \leq C(u)H^{2s}\widetilde{\mathrm{Log}}^s(H/h),$$

$$\|\bar{\partial}\tilde{\rho}^j\| = \|\frac{\tilde{\rho}^j - \tilde{\rho}^{j-1}}{k}\| = \|k^{-1}\int_{t_{j-1}}^{t_j}\tilde{\rho}_t\, d\zeta\| \leq C(u)H^{2s}\widetilde{\mathrm{Log}}^s(H/h),$$

and

$$\|\bar{\partial}u^j - u_t^j\| = \|k^{-1}(u^j - u^{j-1} - ku_t^j)\|$$

$$= \|k^{-1}\int_{t_{j-1}}^{t_j}(\zeta - t_{j-1})u_{tt}(\zeta)\, d\zeta\| \leq C(u)k.$$

Altogether these estimates now yields

$$Q_j = \|\bar{\partial}\tilde{\rho}^j\|^2 + \|\tilde{\rho}^j\|^2 + \|\bar{\partial}u^j - u_t^j\|^2$$

$$\leq C(u)(H^{2s}\widetilde{\mathrm{Log}}^s(H/h) + k)^2, \tag{5.20}$$

which combine with (5.15) and (5.19) gives

$$\|\tilde{\theta}^n\| \leq C\|u_0^{\mathrm{CFE}} - u_0\| + C(u)(H^{2s}\widetilde{\mathrm{Log}}^s(H/h) + k). \tag{5.21}$$

Now, equation (5.21) and Lemma 5.2.2 completes the rest of the proof. □

Next we turn to the uniqueness of the solution of (5.16).

Uniqueness: Given U^{n-1}, let $U^n = P$ and $U^n = Q$ be two solution of (5.16). Further, let $\|u_0^{\mathrm{CFE}} - u_0\| \leq CH^{2s}\widetilde{\mathrm{Log}}^s(H/h)\|u_0\|_{1+s,\Omega}$ holds true. Since both P and Q satisfy (5.16), by subtraction we have

$$(P-Q,\chi) + k(a(P)\nabla P - a(Q)\nabla Q, \nabla\chi) = k(f(P) - f(Q),\chi) \quad \forall\chi \in S^{\mathrm{CFE}}.$$

Setting $\chi = P - Q$, we have

$$\|P-Q\|^2 + k(a(Q)\nabla(P-Q), \nabla(P-Q)) = k(f(P)-f(Q), P-Q) \\ - k((a(P)-a(Q))\nabla P, \nabla(P-Q)),$$

a kickback of the term $\|\nabla(P-Q)\|$ leads to

$$\|P-Q\|^2 + \frac{1}{2}k\widehat{\mu}\|\nabla(P-Q)\|^2 \le C\|P-Q\|^2(k + k\|\nabla P\|^2). \tag{5.22}$$

We now claim that $k\|\nabla P\|^2$ is bounded by an arbitrary small constant, provided $H^2\widetilde{\text{Log}}(H/h)$ and k are sufficiently small. By (5.13) we have

$$\|\nabla P\| \le \|\nabla(U^n - \tilde{U}^n)\| + \|\nabla\tilde{U}^n\| \le \|\nabla\tilde{\theta}^n\| + C(u).$$

Using (5.18), (5.20) and (5.21), we get

$$k\|\nabla\tilde{\theta}^n\|^2 \le C(\|\tilde{\theta}^{n-1}\|^2 + k\|\tilde{\theta}^n\|^2 + kQ_n) \le C(u)(H^{2s}\widetilde{\text{Log}}^s(H/h) + k)^2,$$

which gives

$$k\|\nabla P\|^2 \le C(u)(k + (H^2\widetilde{\text{Log}}(H/h))^{2s}), \tag{5.23}$$

and this proves the claim. Now combining (5.23) with (5.22) we conclude that $\|P-Q\| = 0$, and this completes the proof of uniqueness.

The disadvantage of the method just described above is that at each time level we had to solve a nonlinear system of algebraic equations because of the presence of two nonlinear terms $a(U^n)$ and $f(U^n)$ in (5.16). To avoid this difficulty we modify (5.16) to a linearized form in the following manner.

Linearized modification for the backward Euler method: Replacing the terms $a(U^n)$ and $f(U^n)$ in (5.16) by $a(U^{n-1})$ and $f(U^{n-1})$, respectively, we obtain

$$(\bar{\partial}U^n, \chi) + (a(U^{n-1})\nabla U^n, \nabla\chi) = (f(U^{n-1}), \chi) \quad \forall \chi \in S^{\text{CFE}},\ t_n \in J, \\ \text{with} \quad U^0 = u_0^{\text{CFE}}. \tag{5.24}$$

The resulting linear systems can be solved for α^n, and this can be expressed as

$$(A^{\text{CFE}} + kB^{\text{CFE}}(\alpha^{n-1}))\alpha^n = A^{\text{CFE}}\alpha^{n-1} + kf^{\text{CFE}}(\alpha^{n-1}) \quad \text{for } t_n \in J,$$

where A^{CFE}, B^{CFE} and f^{CFE} are given as before.

We now derive error estimate for the linearized backward Euler CFE method in the following theorem.

Theorem 5.2.3. *Let U^n and u be the solutions of (5.24) and (5.1), respectively. Then under the assumption (5.2) we have,*

$$\|U^n - u(t_n)\| \leq C\|u_0^{CFE} - u_0\| + C(u)(H^{2s}\widetilde{Log}^s(H/h) + k),$$

for $t_n \in \bar{J}$, $1/2 \leq s \leq 1$.

Proof. In view of (5.17) and Lemma 5.2.2, we only need to estimate $\tilde{\theta}^n$. This time we have

$$\begin{aligned}(\bar{\partial}\tilde{\theta}^n, \chi) + (a(U^{n-1})\nabla\tilde{\theta}^n, \nabla\chi) &= (f(U^{n-1}) - f(u^n), \chi) - (\bar{\partial}\tilde{\rho}^n, \chi) - (\bar{\partial}u^n - u_t^n, \chi) \\ &\quad - ((a(U^{n-1}) - a(u^n))\nabla\tilde{U}^n, \nabla\chi),\end{aligned} \tag{5.25}$$

where

$$\|U^{n-1} - u^n\| \leq \|\tilde{\theta}^{n-1}\| + \|\tilde{\rho}^{n-1}\| + k\|\bar{\partial}u^n\|.$$

Setting $\chi = \tilde{\theta}^n$ in (5.25), we obtain

$$\frac{1}{2}\bar{\partial}\|\tilde{\theta}^n\|^2 + \widehat{\mu}\|\nabla\tilde{\theta}^n\|^2 \leq C\|U^{n-1} - u^n\|(\|\tilde{\theta}^n\| + \|\nabla\tilde{\theta}^n\|) + (\|\bar{\partial}\tilde{\rho}^n\| + \|\bar{\partial}u^n - u_t^n\|)\|\tilde{\theta}^n\|.$$

Using the Poincaré's inequality $\|\tilde{\theta}^n\| \leq C\|\nabla\tilde{\theta}^n\|$, we have

$$\frac{1}{2}\bar{\partial}\|\tilde{\theta}^n\|^2 + \widehat{\mu}\|\nabla\tilde{\theta}^n\|^2 \leq C(\|\tilde{\theta}^{n-1}\| + \|\tilde{\rho}^{n-1}\| + k\|\bar{\partial}u^n\| + \|\bar{\partial}\tilde{\rho}^n\| + \|\bar{\partial}u^n - u_t^n\|)\|\nabla\tilde{\theta}^n\|.$$

A similar calculation as before leads to

$$\bar{\partial}\|\tilde{\theta}^n\|^2 \leq C\|\tilde{\theta}^{n-1}\|^2 + C(u)(H^{2s}\widetilde{\text{Log}}^s(H/h) + k)^2,$$

and hence

$$\|\tilde{\theta}^n\|^2 \leq (1 + Ck)\|\tilde{\theta}^{n-1}\|^2 + C(u)k(H^{2s}\widetilde{\text{Log}}^s(H/h) + k)^2.$$

A repeated application yields

$$\|\tilde{\theta}^n\|^2 \leq C\|\tilde{\theta}^0\|^2 + C(u)(H^{2s}\widetilde{\text{Log}}^s(H/h) + k)^2,$$

together with (5.15) gives

$$\|\tilde{\theta}^n\| \leq C\|u_0^{\text{CFE}} - u_0\| + C(u)(H^{2s}\widetilde{\text{Log}}^s(H/h) + k).$$

This completes the proof. □

5.3 Concluding Remarks

This chapter considers a two-scale CFEs for the discretization of nonlinear parabolic problem with Dirichlet boundary conditions in a nonconvex domain. We have studied the error analysis for both the spatially semidiscrete and fully discrete CFE methods for a two-scale CFE discretization. For the fully discrete method, we have used the backward Euler method for the discretization in time of the spatially semidiscrete problem (5.3). We have also studied the modified linearized backward Euler method in order to overcome the difficulty for the backward Euler method (nonlinearized form) in which we need to solve a nonlinear system of algebraic equations at each time level. For the spatially semidiscrete scheme, we use energy method to derive error estimate of order $\mathcal{O}(H^{2s}\widetilde{\mathrm{Log}}^{s}(H/h))$, $1/2 \leq s \leq 1$ in the $L^\infty(L^2)$-norm. The estimates for elliptic projection (see, Lemmas 5.2.2 and 5.2.3) play a key technical tool in our analysis. For the fully discrete scheme, an error estimate of order $\mathcal{O}(H^{2s}\widetilde{\mathrm{Log}}^{s}(H/h)+k)$, $1/2 \leq s \leq 1$ is established. In contrast to the standard finite element procedure, the proposed CFE method not only reduces the dimension of the finite element space but, it is also advantageous for a domain with complicated boundary. Furthermore, the spatially semidiscrete and fully discrete error bounds (Theorems 5.2.1, 5.2.2 and 5.2.3) do not pose any restriction on the fine-scale parameter h. To support our theoretical findings, numerical experiment for a test problem is presented in Chapter 7 (see Example 7.5).

6

Two-Scale CFE Method for Parabolic Problems with Measure Data in Time for Convex and Nonconvex Domains

In this chapter, we study the spatially semidiscrete and fully discrete CFE approximations of parabolic problem (1.4) with measure data in time for both convex and nonconvex domains. The solution of this kind of problem exhibits low regularity with respect to time variable which introduces some difficulties in both theoretical and numerical analysis of FEM. We use piecewise linear and continuous CFEs for the space discretization and derive *a priori* error estimates in the $L^2(L^2)$-norm for the spatially semidiscrete problems, and use the backward Euler method for time discretization and derive *a priori* error estimates in the $L^2(L^2)$-norm for the fully discrete problems. The key technical tools in our error analysis include the estimates for the associated elliptic projection in the framework of CFE method (see Lemmas 2.4.1 and 4.2.1) and the duality trick.

6.1 Introduction

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with boundary Γ, but the boundary of the domain may still have a very complicated shape. Set $\Omega_T := \Omega \times J$ and $\Gamma_T := \Gamma \times J$ with $J := (0, T]$, $T < \infty$. We now recall the following model IBVP:

$$
\begin{aligned}
u_t - \Delta u &= \mu \quad \text{in } \Omega_T, \\
u &= 0 \quad \text{on } \Gamma_T, \\
u(\cdot, 0) &= u_0 \quad \text{in } \Omega,
\end{aligned}
\tag{6.1}
$$

where $u = u(x,t)$, $u_t = \frac{\partial u}{\partial t}$, $\Delta = \sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2}$ the Laplacian and the initial data $u_0 \in L^2(\Omega)$.

Let $\mathcal{M}[0,T]$ be the space of real and regular Borel measures in $[0,T]$ with the norm

$$\|\mu\|_{\mathcal{M}[0,T]} = \sup\left\{\int_0^T v d\mu \ : \ v \in C[0,T] \ \text{ and } \ \|v\|_{C[0,T]} \le 1\right\}.$$

The space $\mathcal{M}[0,T]$ can be defined as the dual space of $C[0,T]$. Thus, μ can be written as $\mu = \sigma\omega$, where σ and ω are given functions such that $\sigma \in C([0,T];L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. We also recall the following spaces:

$$\begin{aligned} W(0,T) &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ X(0,T) &:= L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \ \text{ and} \\ \overline{X}(0,T) &:= L^2(0,T;H^{1+s}(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \quad \text{for } \ 1/2 \le s \le 1. \end{aligned}$$

It is straightforward that (cf. [66])

$$W(0,T) \hookrightarrow C([0,T];L^2(\Omega)) \quad \text{and} \quad X(0,T),\ \overline{X}(0,T) \hookrightarrow C([0,T];H_0^1(\Omega)).$$

The weak formulation of the problem (6.1) may be stated as follows by using transposition techniques (cf. [66]): Find $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ such that

$$\begin{aligned} -(u,v_t) + \mathcal{A}(u,v)_{\Omega_T} &= \langle \mu, v\rangle_{\Omega_T} + (u_0, v(\cdot,0)) \quad \forall v \in W(0,T), \\ \text{with} \quad v(\cdot,T) &= 0, \end{aligned} \tag{6.2}$$

where $\mathcal{A}(u,v)_{\Omega_T} = (\nabla u, \nabla v)_{\Omega_T}$ and

$$\langle \mu, v\rangle_{\Omega_T} = \int_{\Omega_T} v d\mu = \int_0^T \left(\int_\Omega \sigma(x,t)v(x,t)dx\right) d\omega(t) \quad \forall v \in C([0,T];L^2(\Omega)).$$

In order to define the CFE space, we first define the continuous and piecewise linear finite element space S corresponding to the triangulation $\mathcal{T}_{H,h}$ as

$$S := \{v \in C^0(\Omega_{H,h}) \ | \ v|_K \in \mathbb{P}_1 \ \ \forall K \in \mathcal{T}_{H,h}\},$$

where $\Omega_{H,h} := \text{int}\left(\bigcup_{K\in\mathcal{T}_{H,h}} K\right)$. The CFE space S^{CFE} corresponding to the triangulation $\mathcal{T}_{H,h}$ is defined by

$$S^{\text{CFE}} := \{v \in S \mid \exists\, \mathbf{\Phi} \in \mathbb{R}^{\vartheta_{\text{dof}}} \ \text{ with } v(x) = (\mathcal{E}\mathbf{\Phi})_x \ \ \forall x \in \vartheta_{H,h}\},$$

where $\mathbf{\Phi} : \vartheta_{\text{dof}} \to \mathbb{R}$ is the grid function (as described in Chapter 1) and the extrapolation operator $\mathcal{E} : \mathbb{R}^{\vartheta_{\text{dof}}} \to \mathbb{R}^{\vartheta_{H,h}}$ is defined in (1.11). Clearly S^{CFE} is a subspace of S.

We now state the spatially semidiscrete CFE approximation based on the weak formulation (6.2) is to find $u^{\text{CFE}}(t) \in L^2(0,T;S^{\text{CFE}})$ such that

$$-(u^{\text{CFE}}, v_t^{\text{CFE}})_{\Omega_T} + \mathcal{A}(u^{\text{CFE}}, v^{\text{CFE}})_{\Omega_T} = \langle \mu, v^{\text{CFE}}\rangle_{\Omega_T} + (u_0^{\text{CFE}}, v^{\text{CFE}}(\cdot,0)) \tag{6.3}$$

$\forall v^{\mathrm{CFE}} \in H^1(0,T;S^{\mathrm{CFE}})$, with $v^{\mathrm{CFE}}(\cdot,T)=0$, where $u_0^{\mathrm{CFE}} \in S^{\mathrm{CFE}}$ is an approximation of u_0 and

$$\langle \mu, v^{\mathrm{CFE}}\rangle_{\Omega_T} = \int_{\overline{\Omega}_T} v^{\mathrm{CFE}} d\mu = \int_0^T \left(\int_\Omega \sigma(x,t) v^{\mathrm{CFE}}(x) dx \right) d\omega(t) \quad \forall v^{\mathrm{CFE}} \in H^1(0,T;S^{\mathrm{CFE}}).$$

The numerical treatment of elliptic and parabolic equations in convex domains by the standard FEM involving measure data are thoroughly studied in Boccardo *et al.* [13], Boccardo and Gallouët [14], Casas [18], Gallouët and Herbin [42] and Gong [46]. In [18], Casas has studied the FEM for second order elliptic problems with homogeneous Dirichlet condition involving measure data in convex domains, where the data are considered as real and regular Borel measures. A convergence of order $\mathcal{O}(h)$ in two-dimensions and $\mathcal{O}(h^{1/2})$ in three dimensions in the L^2-norm have been established. Recently, error estimates of FEMs for parabolic equations with measure data in a domain with smooth boundary have been investigated by Gong in [46]. Due to presence of measure data in time the solution of such problems possess less regularity and hence the standard finite element error analysis tricks do not apply directly. However, writing the weak formulation by transposition techniques (cf. [66]), the author of [46] has proved the existence and uniqueness of problem (6.1) [46, Theorem 2.2]. *A priori* error estimates for both spatially semidiscrete and fully discrete problems have been analyzed and optimal error estimates of order $\mathcal{O}(h)$ for the space discretization and $\mathcal{O}(k^{1/2})$ for the time discretization have been established.

In this chapter, we investigate the convergence analysis of the CFE approximations of parabolic problem (6.1) with measure data in time for both convex and nonconvex polygonal domains. *A priori* error estimates in the $L^2(L^2)$-norm for the spatially semidiscrete and fully discrete problems for both convex and nonconvex domains are derived.

The structure of the remaining sections of this chapter is as follows. In Section 6.2, we present some auxiliary results which will be used in the error analysis. The spatially semidiscrete and fully discrete error estimates for the domain with smooth boundary, or a convex polygonal domain are derived in Section 6.3. Section 6.4 is devoted to the error analysis for nonconvex polygonal domains for both spatially semidiscrete and fully discrete schemes. The fully discretization error analysis is based on the backward Euler method in both convex and nonconvex domains. Finally, we present some concluding remarks in Section 6.5.

6.2 Some Auxiliary Results

In this section, we shall state the existence and uniqueness result for the solution of (6.1) and some auxiliary estimates which will be used in the error analysis. Let ξ be the solution of the forward in time parabolic problem given by, for $f \in L^2(\Omega_T)$,

$$\begin{aligned} \xi_t - \Delta\xi &= f \quad \text{in} \quad \Omega_T, \\ \xi &= 0 \quad \text{on} \quad \Gamma_T, \\ \xi(0) &= 0 \quad \text{in} \quad \Omega. \end{aligned} \tag{6.4}$$

Also, let η be the solution of the following backward in time parabolic problem, for $f \in L^2(\Omega_T)$,

$$\begin{aligned} -\eta_t - \Delta\eta &= f \quad \text{in} \quad \Omega_T, \\ \eta &= 0 \quad \text{on} \quad \Gamma_T, \\ \eta(T) &= 0 \quad \text{in} \quad \Omega. \end{aligned} \tag{6.5}$$

Following [70], we have the standard stability estimates for the solutions of (6.4) and (6.5).

Lemma 6.2.1. *Let ξ and η be the solutions of problem (6.4) and (6.5), respectively. Assume $z = \xi$ or $z = \eta$, then $z \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \hookrightarrow C([0,T];H_0^1(\Omega))$ and satisfies*

$$\|z\|_{L^2(0,T;H^2(\Omega))} + \|z_t\|_{L^2(0,T;L^2(\Omega))} \leq C\|f\|_{L^2(0,T;L^2(\Omega))}.$$

The next theorem shows the existence, uniqueness and regularity of the solution of (6.1). For a proof, see [46].

Theorem 6.2.1. *Let $\mu = \sigma\omega$, where σ and ω are given functions such that $\sigma \in C([0,T];L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$, and define the following inner product*

$$\langle \mu, v\rangle_{\Omega_T} = \int_{\overline{\Omega}_T} v d\mu = \int_0^T \left(\int_\Omega \sigma(x,t)v(x,t)dx \right) d\omega(t) \quad \forall v \in C([0,T];L^2(\Omega)).$$

Then there exists a unique solution $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ of the problem (6.1) such that

$$\begin{aligned} -(u, v_t) + \mathcal{A}(u,v)_{\Omega_T} &= \langle \mu, v\rangle_{\Omega_T} + (u_0, v(\cdot,0)) \quad \forall v \in W(0,T), \\ v(\cdot,T) &= 0, \end{aligned}$$

where $\mathcal{A}(u,v)_{\Omega_T} = (\nabla u, \nabla v)_{\Omega_T}$. *Further, we have*

$$\|u\|_{L^2(0,T;H_0^1(\Omega))} + \|u\|_{L^\infty(0,T;L^2(\Omega))} \le C(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|u_0\|_{0,\Omega}).$$

For the purpose of error analysis of the semidiscrete problem, we introduce so-called *elliptic* or *Ritz projection* R^{CFE} onto S^{CFE} as the orthogonal projection with respect to the inner product $(\nabla v, \nabla w)$ so that

$$(\nabla R^{\mathrm{CFE}} v, \nabla \chi) = (\nabla v, \nabla \chi) \quad \forall \chi \in S^{\mathrm{CFE}}, \ \text{ for } v \in H_0^1(\Omega). \tag{6.6}$$

It is easy to verify the stability of the elliptic projection in H^1-norm, i.e.,

$$\|\nabla R^{\mathrm{CFE}} v\| \le \|\nabla v\| \quad \forall v \in H_0^1(\Omega).$$

For convenience, we now recall the following auxiliary results for the errors in the Ritz projection for convex and nonconvex domains, respectively (cf. Lemmas 2.4.1 and 4.2.1).

Lemma 6.2.2. *Let* $v \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* R^{CFE} *be defined by (6.6). Assume that the conditions (1.9) and (2.9) hold true. Then, we have*

$$\|\nabla(R^{CFE}v - v)\| \le CH\widetilde{Log}^{1/2}(H/h)\|v\|_{2,\Omega},$$

and

$$\|R^{CFE}v - v\| \le CH^2\widetilde{Log}(H/h)\|v\|_{2,\Omega}.$$

where the constant C *depends on* σ_{dist}, σ_e, σ_{uni}, $C_{ol,1}$, $C_{ol,2}$, σ_{ext} *and the minimal angles in the triangulation* $\mathcal{T}_{H,h}$.

Lemma 6.2.3. *Let* $v \in H_0^1(\Omega) \cap H^{1+s}(\Omega)$ *for* $1/2 \le s \le 1$ *and* R^{CFE} *be defined by (6.6). Assume that the conditions (1.9) and (2.9) hold true. Then there exists a positive constant* C *which depends on* σ_{dist}, σ_e, σ_{uni}, $C_{ol,1}$, $C_{ol,2}$, σ_{ext} *and the minimal angles in the triangulation* $\mathcal{T}_{H,h}$, *such that*

$$\|\nabla(R^{CFE}v - v)\| \le CH^s\widetilde{Log}^{s/2}(H/h)\|v\|_{1+s,\Omega},$$

and

$$\|R^{CFE}v - v\| \le CH^{2s}\widetilde{Log}^{s}(H/h)\|v\|_{1+s,\Omega},$$

for $1/2 \le s \le 1$.

6.3 CFE Error Estimates for Convex Domains

To begin with, we first write the CFE approximation of the forward parabolic problem (6.4) as follows: Find $\xi^{\text{CFE}}(t) \in H^1(0,T;S^{\text{CFE}})$ such that

$$
\begin{aligned}
(\xi_t^{\text{CFE}},\chi)_{\Omega_T} + \mathcal{A}(\xi^{\text{CFE}},\chi)_{\Omega_T} &= (f,\chi)_{\Omega_T} \quad \forall \chi \in S^{\text{CFE}}, \\
\text{with} \quad (\xi^{\text{CFE}}(0),\chi) &= 0 \quad \forall \chi \in S^{\text{CFE}}.
\end{aligned} \tag{6.7}
$$

Similarly, we now pose the CFE approximation of the backward parabolic problem (6.5): Find $\eta^{\text{CFE}}(t) \in H^1(0,T;S^{\text{CFE}})$ such that

$$
\begin{aligned}
-(\eta_t^{\text{CFE}},\chi)_{\Omega_T} + \mathcal{A}(\eta^{\text{CFE}},\chi)_{\Omega_T} &= (f,\chi)_{\Omega_T} \quad \forall \chi \in S^{\text{CFE}}, \\
\text{with} \quad (\eta^{\text{CFE}}(T),\chi) &= 0 \quad \forall \chi \in S^{\text{CFE}}.
\end{aligned} \tag{6.8}
$$

6.3.1 Spatially Semidiscrete Error Estimates

Now we concentrate on the spatially semidiscrete error estimates of the problem (6.1) with measure data in time. we start with the following auxiliary result.

Lemma 6.3.1. *Let ξ and η denote the solutions of problem (6.4) and (6.5), respectively. Also, let ξ^{CFE} and η^{CFE} be the solutions of (6.7) and (6.8), respectively, where $\xi,\eta \in X(0,T) \hookrightarrow C([0,T];H_0^1(\Omega))$ and $\xi^{CFE},\eta^{CFE} \in H^1(0,T;S^{CFE})$. Then for $\tilde{e} = \xi - \xi^{CFE}$ or $\tilde{e} = \eta - \eta^{CFE}$, we have*

$$
\|\tilde{e}\|_{L^\infty(0,T;L^2(\Omega))} \le CH\widetilde{Log}^{1/2}(H/h)\|f\|_{L^2(0,T;L^2(\Omega))}. \tag{6.9}
$$

Proof. With $\tilde{e}(t) = \xi(t) - \xi^{\text{CFE}}(t)$, and using the variational form of (6.4) and equation (6.7), we write the error identity

$$
(\tilde{e}_t,\chi) + \mathcal{A}(\tilde{e},\chi) = 0 \quad \forall \chi \in S^{\text{CFE}}.
$$

Setting $\chi = R^{\text{CFE}}\tilde{e}$, this yields

$$
(\tilde{e}_t, R^{\text{CFE}}\tilde{e}) + \mathcal{A}(\tilde{e}, R^{\text{CFE}}\tilde{e}) = 0,
$$

which implies

$$
\begin{aligned}
(\tilde{e}_t,\tilde{e}) + \mathcal{A}(\tilde{e},\tilde{e}) &= (\tilde{e}_t, \xi - R^{\text{CFE}}\xi) + \mathcal{A}(\tilde{e}, \xi - R^{\text{CFE}}\xi) \\
&\le \|\tilde{e}_t\|\|R^{\text{CFE}}\xi - \xi\| + C\|\tilde{e}\|_1\|R^{\text{CFE}}\xi - \xi\|_1.
\end{aligned} \tag{6.10}
$$

Lemma 6.2.2 implies

$$\frac{d}{dt}\|\tilde{e}\|^2 + C_0\|\tilde{e}\|_1^2 \leq CH^2\widetilde{\mathrm{Log}}(H/h)(\|\tilde{e}_t\|^2 + \|\xi\|_2^2),$$

for some positive constant C_0. Integrating the above from 0 to t we obtain

$$\|\tilde{e}(t)\|^2 + C_0\int_0^t \|\tilde{e}\|_1^2 d\zeta \leq CH^2\widetilde{\mathrm{Log}}(H/h)\int_0^t (\|\tilde{e}_\zeta\|^2 + \|\xi\|_2^2)d\zeta. \tag{6.11}$$

Analogous to Lemma 6.2.1, we have the following estimate on the discrete level

$$\|\xi_t^{\mathrm{CFE}}\|_{L^2(0,T;L^2(\Omega))} \leq C\|f\|_{L^2(0,T;L^2(\Omega))},$$

which combine with Lemma 6.2.1 yields

$$\int_0^t \|\tilde{e}_\zeta\|^2 d\zeta = \int_0^t \|\xi_\zeta - \xi_\zeta^{\mathrm{CFE}}\|^2 d\zeta \leq C\int_0^T \|f\|^2 d\zeta. \tag{6.12}$$

Now, using (6.12) and Lemma 6.2.1, it follows from (6.11) that

$$\|\xi(t) - \xi^{\mathrm{CFE}}(t)\|^2 + C_0\int_0^t \|\xi - \xi^{\mathrm{CFE}}\|_1^2 d\zeta \leq CH^2\widetilde{\mathrm{Log}}(H/h)\int_0^T \|f\|^2 d\zeta \quad \forall t \in [0,T],$$

and this shows (6.9).

Arguing in a similar way, for $\tilde{e} = \eta - \eta^{\mathrm{CFE}}$ with $\tilde{e}(T) = 0$, we obtain

$$\|\eta(t) - \eta^{\mathrm{CFE}}(t)\|^2 + C_0\int_0^t \|\eta - \eta^{\mathrm{CFE}}\|_1^2 d\zeta \leq CH^2\widetilde{\mathrm{Log}}(H/h)\int_0^T \|f\|^2 d\zeta, \tag{6.13}$$

$\forall t \in [0,T]$ and this completes the proof. □

We now concentrate on the error estimates between the solutions of the continuous problem (6.2) and the semidiscrete problem (6.3). As the solution of (6.2) has low regularity it is, therefore, reasonable to estimate the error in the $L^2(L^2)$-norm. Letting P^{CFE} be the L^2-projection onto S^{CFE}, we set $u_0^{\mathrm{CFE}} = P^{\mathrm{CFE}}u_0$ be the L^2-projection of u_0.

Theorem 6.3.1. *Let $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and $u^{CFE} \in L^2(0,T;S^{CFE})$ be the solutions of the problems (6.2) and (6.3), respectively, with $\mu = \sigma\omega$, σ and ω are given functions such that $\sigma \in C([0,T];L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. Then, we have*

$$\|u - u^{CFE}\|_{L^2(0,T;L^2(\Omega))} \leq CH\widetilde{Log}^{1/2}(H/h)(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|u_0\|_{0,\Omega}).$$

Proof. Let η be the solution of the problem (6.5) with $f \in L^2(0,T;L^2(\Omega))$. Then using the continuous problem (6.2) and the semidiscrete problem (6.3) together with the orthogonality property, we have

$$\begin{aligned}
\int_{\Omega_T} (u-u^{\text{CFE}}) f dxdt \\
&= \int_0^T \int_\Omega (u-u^{\text{CFE}})(-\eta_t - \Delta\eta) dxdt \\
&= (u, -\eta_t)_{\Omega_T} + \mathcal{A}(u,\eta)_{\Omega_T} + (u^{\text{CFE}}, \eta_t)_{\Omega_T} - \mathcal{A}(u^{\text{CFE}}, \eta)_{\Omega_T} \\
&= \langle \mu, \eta \rangle_{\Omega_T} + (u_0, \eta(0)) + (u^{\text{CFE}}, \eta_t^{\text{CFE}})_{\Omega_T} - \mathcal{A}(u^{\text{CFE}}, \eta^{\text{CFE}})_{\Omega_T} \\
&= \langle \mu, \eta \rangle_{\Omega_T} + (u_0, \eta(0)) - \langle \mu, \eta^{\text{CFE}} \rangle_{\Omega_T} - (P^{\text{CFE}} u_0, \eta^{\text{CFE}}(0)) \\
&= \langle \mu, \eta - \eta^{\text{CFE}} \rangle_{\Omega_T} + (u_0, \eta(0) - \eta^{\text{CFE}}(0)) \\
&= \int_0^T \left(\int_\Omega \sigma(x,t)(\eta - \eta^{\text{CFE}}) dx \right) d\omega(t) + (u_0, \eta(0) - \eta^{\text{CFE}}(0)). \quad (6.14)
\end{aligned}$$

Now using the relation $\|\omega\|_{L^1[0,T]} \leq \|\omega\|_{\mathcal{M}[0,T]}$, the first term on the right-hand side of (6.14) gives

$$\begin{aligned}
\int_0^T \left(\int_\Omega \sigma(x,t)(\eta - \eta^{\text{CFE}}) dx \right) d\omega(t) \\
&\leq \int_0^T \|\sigma(t)\|_{L^2(\Omega)} \|\eta(t) - \eta^{\text{CFE}}(t)\|_{L^2(\Omega)} d\omega(t) \\
&\leq \|\sigma\|_{L^\infty(0,T;L^2(\Omega))} \|\omega\|_{L^1[0,T]} \|\eta - \eta^{\text{CFE}}\|_{L^\infty(0,T;L^2(\Omega))} \\
&\leq \|\sigma\|_{L^\infty(0,T;L^2(\Omega))} \|\omega\|_{\mathcal{M}[0,T]} \|\eta - \eta^{\text{CFE}}\|_{L^\infty(0,T;L^2(\Omega))},
\end{aligned}$$

and hence, (6.14) yields

$$\begin{aligned}
\int_{\Omega_T} (u-u^{\text{CFE}}) f dxdt \leq C(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))} \|\omega\|_{\mathcal{M}[0,T]} \\
+ \|u_0\|_{0,\Omega}) \|\eta - \eta^{\text{CFE}}\|_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}$$

An use of Lemma 6.3.1 leads to

$$\begin{aligned}
\int_{\Omega_T} (u-u^{\text{CFE}}) f dxdt \leq CH\widetilde{\text{Log}}^{1/2}(H/h)(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))} \|\omega\|_{\mathcal{M}[0,T]} \\
+ \|u_0\|_{0,\Omega}) \|f\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

and by the definition of $L^2(\Omega_T)$-norm, we obtain

$$\begin{aligned}\|u - u^{\text{CFE}}\|_{L^2(0,T;L^2(\Omega))} &= \sup_{\substack{f \in L^2(0,T;L^2(\Omega)) \\ f \neq 0}} \frac{(f, u - u^{\text{CFE}})_{\Omega_T}}{\|f\|_{L^2(0,T;L^2(\Omega))}} \\ &\leq CH\widetilde{\text{Log}}^{1/2}(H/h)(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|u_0\|_{0,\Omega}),\end{aligned}$$

which completes the proof of the theorem. □

Remark 6.3.1. *From Theorem 6.3.1 it is clear that, there is no restriction on the fine-scale parameter h. In fact, the results of the theorem also holds when the two-scale grid $\mathcal{T}_{H,h}$ coincide with the coarse-scale grid $\mathcal{T}_H$, i.e., $h = \mathcal{O}(H)$. When $h = \mathcal{O}(H)$, Theorem 6.3.1 yields the same ROC as in the standard FEM (cf. Gong [46]).*

6.3.2 Fully Discrete Error Estimates

In this section, we turn our attention to the fully discrete approximations for the semidiscrete problem (6.3). As a fully discrete approximation we consider the *backward Euler* scheme for the time discretization. Let k be the time step and consider the time points

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T.$$

Let $\bar{J} = [0, T]$ be the time interval partitioning as

$$\bar{J} = \{0\} \cup J_1 \cup J_2 \cup \cdots \cup J_N,$$

with subintervals $J_i = (t_{i-1}, t_i]$ of size k. Replacing the time derivative in (6.3) by the backward Euler quotient to find $U_i^{\text{CFE}} \in S^{\text{CFE}}$, $i = 1, 2, \cdots, N$ such that

$$\begin{aligned}(\frac{U_i^{\text{CFE}} - U_{i-1}^{\text{CFE}}}{k}, v^{\text{CFE}}) + \mathcal{A}(U_i^{\text{CFE}}, v^{\text{CFE}}) &= \langle \mu, v^{\text{CFE}} \rangle_{J_i} \quad \forall v^{\text{CFE}} \in S^{\text{CFE}}, \\ \text{with } U_0^{\text{CFE}}(x) &= u_0^{\text{CFE}}(x), \quad x \in \Omega,\end{aligned} \tag{6.15}$$

where U^{CFE} denotes the fully discrete approximation of u in S^{CFE}. U^{CFE} is piecewise constant in time and piecewise linear in space on each time interval. We set $k = \mathcal{O}(h^2)$ in our analysis. Here

$$\langle \mu, v^{\text{CFE}} \rangle_{J_i} = \frac{1}{k} \int_{\Omega \times (t_{i-1}, t_i]} v^{\text{CFE}} d\mu = \frac{1}{k} \int_{t_{i-1}}^{t_i} \left(\int_\Omega \sigma(x,t) v^{\text{CFE}}(x) dx \right) d\omega(t) \quad \forall v^{\text{CFE}} \in S^{\text{CFE}}.$$

In the following, we derive stability estimate for the solution of (6.15).

Lemma 6.3.2. *Let $U_i^{CFE} \in S^{CFE}$, $i = 1, 2, \cdots, N$ be the solutions of the fully discrete scheme (6.15). Assume that $u_0^{CFE} = P^{CFE} u_0$ and $k \le Ch^2$, then we have*

$$\sum_{i=1}^{N} \|U_i^{CFE} - U_{i-1}^{CFE}\|_{0,\Omega}^2 + k\|U_N^{CFE}\|_{1,\Omega}^2 \le C(\|\sigma\|^2_{L^\infty(0,T;L^2(\Omega))}\|\omega\|^2_{\mathcal{M}[0,T]} + \|u_0\|^2_{0,\Omega}),$$

where C is a constant independent of $(h,\ H,\ k)$ and the data $(\sigma,\ \omega,\ u_0)$.

Proof. The proof follows the idea of [46]. Setting $v^{\mathrm{CFE}} = k(U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}})$ in (6.15), we have

$$(U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}, U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}) + k\mathcal{A}(U_i^{\mathrm{CFE}}, U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}) = k\langle \mu, U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}\rangle_{J_i},$$

or,

$$\begin{aligned}
&\|U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}\|_{0,\Omega}^2 + k\|U_i^{\mathrm{CFE}}\|_{1,\Omega}^2 \\
&\le k\mathcal{A}(U_i^{\mathrm{CFE}}, U_{i-1}^{\mathrm{CFE}}) + \int_{t_{i-1}}^{t_i} (\sigma(t), U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}) d\omega(t) \\
&\le \frac{1}{2}k\|U_i^{\mathrm{CFE}}\|_{1,\Omega}^2 + \frac{1}{2}k\|U_{i-1}^{\mathrm{CFE}}\|_{1,\Omega}^2 + C\Big(\int_{t_{i-1}}^{t_i} \|\sigma(t)\|_{0,\Omega} d\omega(t)\Big)^2 + \frac{1}{2}\|U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}\|_{0,\Omega}^2.
\end{aligned}$$

Summing over i from 1 to N, this yields

$$\begin{aligned}
\sum_{i=1}^{N} \|U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}\|_{0,\Omega}^2 \quad &+ \quad k\|U_N^{\mathrm{CFE}}\|_{1,\Omega}^2 \\
&\le \quad k\|P^{\mathrm{CFE}}u_0\|_{1,\Omega}^2 + C\sum_{i=1}^{N}\Big(\int_{t_{i-1}}^{t_i} \|\sigma(t)\|_{0,\Omega} d\omega(t)\Big)^2 \\
&\le \quad k\|P^{\mathrm{CFE}}u_0\|_{1,\Omega}^2 + C\Big(\int_0^T \|\sigma(t)\|_{0,\Omega} d\omega(t)\Big)^2 \\
&\le \quad k\|P^{\mathrm{CFE}}u_0\|_{1,\Omega}^2 + C\|\sigma\|^2_{L^\infty(0,T;L^2(\Omega))}\|\omega\|^2_{\mathcal{M}[0,T]}. \qquad (6.16)
\end{aligned}$$

Now for any $\tau \in \mathrm{sons}(K)$, using the inverse estimate (see e.g., [16, Section 4.5]) and the fact $k \le Ch^2 \le Ch_\tau^2$, we obtain

$$k\|P_\tau^{\mathrm{CFE}}u_0\|_{1,\tau}^2 \le kh_\tau^{-2}\|P_\tau^{\mathrm{CFE}}u_0\|_{0,\tau}^2 \le C\|u_0\|_{0,\tau}^2,$$

where $P_\tau^{\mathrm{CFE}}u_0$ is the L^2-projection of u_0 on τ, and hence

$$\begin{aligned}
k\|P^{\mathrm{CFE}}u_0\|_{1,\Omega}^2 \quad &\le \quad \sum_{K\in\mathcal{T}_H}\sum_{\tau\in\mathrm{sons}(K)} k\|P_\tau^{\mathrm{CFE}}u_0\|_{1,\tau}^2 \\
&\le \quad C\sum_{K\in\mathcal{T}_H}\sum_{\tau\in\mathrm{sons}(K)} \|u_0\|_{0,\tau}^2 \le C\|u_0\|_{0,\Omega}^2.
\end{aligned}$$

Therefore, (6.16) gives

$$\sum_{i=1}^{N} \|U_i^{\text{CFE}} - U_{i-1}^{\text{CFE}}\|_{0,\Omega}^2 + k\|U_N^{\text{CFE}}\|_{1,\Omega}^2 \leq C(\|u_0\|_{0,\Omega}^2 + \|\sigma\|_{L^\infty(0,T;L^2(\Omega))}^2 \|\omega\|_{\mathcal{M}[0,T]}^2).$$

This proves the desired result. □

Now we are ready to prove the main result. The following theorem gives the error estimate in the $L^2(L^2)$-norm between the solution of the continuous problem (6.2) and the fully discrete problem (6.15).

Theorem 6.3.2. *Assume $\mu = \sigma\omega$, σ and ω are given functions such that $\sigma \in C([0,T]; L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. Let $u \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ and U^{CFE} be the solution of (6.2) and (6.15), respectively. Then, we have*

$$\begin{aligned}
&\|u - U^{CFE}\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad \leq C(H\widetilde{Log}^{1/2}(H/h) + k^{1/2})(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|u_0\|_{0,\Omega}).
\end{aligned}$$

Proof. Let η be the solution of (6.5) and $f \in L^2(0,T; L^2(\Omega))$. In analogy with (6.2) we write

$$\begin{aligned}
&\int_{\Omega_T} (u - U^{\text{CFE}}) f\,dxdt \\
&\quad = \int_0^T \int_\Omega (u - U^{\text{CFE}})(-\eta_t - \Delta\eta)\,dxdt \\
&\quad = -(u, \eta_t)_{\Omega_T} + \mathcal{A}(u,\eta)_{\Omega_T} + \sum_{i=1}^{N} \int_{J_i} \{k^{-1}(U_i^{\text{CFE}}, \eta^i - \eta^{i-1}) - \mathcal{A}(U_i^{\text{CFE}}, \eta)\}dt \\
&\quad = \langle \mu, \eta\rangle_{\Omega_T} + (u_0, \eta(0)) - \sum_{i=1}^{N} \int_{J_i} \{k^{-1}(U_i^{\text{CFE}} - U_{i-1}^{\text{CFE}}, \eta^{i-1}) + \mathcal{A}(U_i^{\text{CFE}}, \eta)\}dt \\
&\qquad\quad + (U_N^{\text{CFE}}, \eta^N) - (U_0^{\text{CFE}}, \eta(0)) \\
&\quad = -\sum_{i=1}^{N} \int_{J_i} \{k^{-1}(U_i^{\text{CFE}} - U_{i-1}^{\text{CFE}}, \eta^{i-1}) + \mathcal{A}(U_i^{\text{CFE}}, \eta)\}dt + \langle \mu, \eta\rangle_{\Omega_T} \\
&\qquad\quad + (u_0 - U_0^{\text{CFE}}, \eta(0)).
\end{aligned}$$

Now from (6.15), we obtain

$$\sum_{i=1}^{N} \{k^{-1}(U_i^{\text{CFE}} - U_{i-1}^{\text{CFE}}, \overline{R^{\text{CFE}}}\eta) + \mathcal{A}(U_i^{\text{CFE}}, \overline{R^{\text{CFE}}}\eta)\} = \sum_{i=1}^{N} \langle \mu, \overline{R^{\text{CFE}}}\eta\rangle_{J_i},$$

where $\overline{R^{\mathrm{CFE}}}\eta \in S^{\mathrm{CFE}}$ is defined on J_i as

$$\overline{R^{\mathrm{CFE}}}\eta = \overline{R^{\mathrm{CFE}}}\eta^i = \overline{R^{\mathrm{CFE}}}\eta(\cdot, t_i) = \frac{1}{k}\int_{J_i} R^{\mathrm{CFE}}\eta(\cdot, t)dt, \quad i > 0, \tag{6.17}$$

with $\overline{R^{\mathrm{CFE}}}\eta^N = R^{\mathrm{CFE}}\eta(T)$. Also, $\bar{\eta}$ denotes the average of η in J_i as in (6.17), and hence it follows that

$$\int_{J_i}(\eta - \bar{\eta})dt = 0.$$

Thus, we find

$$\begin{aligned}
\int_{\Omega_T}(u - U^{\mathrm{CFE}})fdxdt &= -\sum_{i=1}^{N}\int_{J_i}\{k^{-1}(U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}, \eta^{i-1}) + \mathcal{A}(U_i^{\mathrm{CFE}}, \bar{\eta})\}dt \\
&+\langle \mu, \eta\rangle_{\Omega_T} - \sum_{i=1}^{N}\int_{J_i}\langle \mu, \overline{R^{\mathrm{CFE}}}\eta\rangle_{J_i} + \sum_{i=1}^{N}\int_{J_i}\Big\{k^{-1}(U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}, \overline{R^{\mathrm{CFE}}}\eta) \\
&+\mathcal{A}(U_i^{\mathrm{CFE}}, \overline{R^{\mathrm{CFE}}}\eta)\Big\}dt + (u_0 - U_0^{\mathrm{CFE}}, \eta(0))
\end{aligned}$$

$$\begin{aligned}
&= -\sum_{i=1}^{N}\int_{J_i}\{k^{-1}(U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}, \eta^{i-1} - \overline{R^{\mathrm{CFE}}}\eta) + \mathcal{A}(U_i^{\mathrm{CFE}}, \bar{\eta} - \overline{R^{\mathrm{CFE}}}\eta)\}dt \\
&+\{\langle \mu, \eta\rangle_{\Omega_T} - \sum_{i=1}^{N}\int_{J_i}\langle \mu, \overline{R^{\mathrm{CFE}}}\eta\rangle_{J_i}\} + (u_0 - U_0^{\mathrm{CFE}}, \eta(0)) := Q_1 + Q_2 + Q_3.
\end{aligned} \tag{6.18}$$

To estimate Q_1 we proceed as follows. From the definition of Ritz projection and for $U_i^{\mathrm{CFE}} \in S^{\mathrm{CFE}}$, we obtain

$$\int_{J_i}\mathcal{A}(U_i^{\mathrm{CFE}}, \bar{\eta} - \overline{R^{\mathrm{CFE}}}\eta)dt = 0,$$

and hence, by the Cauchy-Schwarz inequality we have

$$\begin{aligned}
|Q_1| &= |-\sum_{i=1}^{N}\int_{J_i}k^{-1}(U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}, \eta^{i-1} - \overline{R^{\mathrm{CFE}}}\eta)dt| \\
&\leq \left(\sum_{i=1}^{N}\|U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}\|_{0,\Omega}^2\right)^{1/2} \cdot \left(\sum_{i=1}^{N}\|\eta^{i-1} - \overline{R^{\mathrm{CFE}}}\eta\|_{0,\Omega}^2\right)^{1/2} := G_1 \,.\, G_2,
\end{aligned} \tag{6.19}$$

where

$$G_1 = \left(\sum_{i=1}^{N}\|U_i^{\mathrm{CFE}} - U_{i-1}^{\mathrm{CFE}}\|_{0,\Omega}^2\right)^{1/2}, \quad G_2 = \left(\sum_{i=1}^{N}\|\eta^{i-1} - \overline{R^{\mathrm{CFE}}}\eta\|_{0,\Omega}^2\right)^{1/2}.$$

In view of Lemma 6.3.2, we obtain

$$G_1 \leq C(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|u_0\|_{0,\Omega}).$$

Now,

$$\|\eta^{i-1} - \overline{R^{\mathrm{CFE}}}\eta\|_{0,\Omega} \leq \|\eta^{i-1} - \bar{\eta}\|_{0,\Omega} + \|\bar{\eta} - \overline{R^{\mathrm{CFE}}}\eta\|_{0,\Omega}.$$

Since

$$\|\eta^{i-1} - \bar{\eta}\|_{0,\Omega} \leq k^{1/2}\|\eta_t\|_{L^2(t_{i-1},t_i;L^2(\Omega))},$$

using Lemma 6.2.2 and the definition of $\bar{\eta}$, we have

$$\|\bar{\eta} - \overline{R^{\mathrm{CFE}}}\eta\|_{0,\Omega} \leq CH^2\widetilde{\mathrm{Log}}(H/h)\|\bar{\eta}\|_{2,\Omega} \leq CH^2\widetilde{\mathrm{Log}}(H/h)k^{-1/2}\|\eta\|_{L^2(t_{i-1},t_i;H^2(\Omega))}.$$

Using Lemma 6.2.1 and the fact $Ch^2 \leq CH^2$, we conclude that

$$\begin{aligned} G_2 &\leq C\left[\sum_{i=1}^{N}\left(k\|\eta_t\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))} + H^4\widetilde{\mathrm{Log}}^2(H/h)k^{-1}\|\eta\|^2_{L^2(t_{i-1},t_i;H^2(\Omega))}\right)\right]^{1/2} \\ &\leq C(H\widetilde{\mathrm{Log}}(H/h) + k^{1/2})\|f\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

Therefore, (6.19) implies that

$$\begin{aligned} |Q_1| \leq C(H\widetilde{\mathrm{Log}}(H/h) + k^{1/2})(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} \\ + \|u_0\|_{0,\Omega})\|f\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{6.20}$$

Furthermore,

$$\begin{aligned} |Q_2| &= |\langle \mu, \eta\rangle_{\Omega_T} - \sum_{i=1}^{N}\int_{J_i}\langle \mu, \overline{R^{\mathrm{CFE}}}\eta\rangle_{J_i}| \\ &= |\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(\int_\Omega \sigma(x,t)(\eta - \overline{R^{\mathrm{CFE}}}\eta)(x)dx)d\omega(t)| \\ &\leq C\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}\|\eta - \overline{R^{\mathrm{CFE}}}\eta\|_{L^\infty(0,T;L^2(\Omega))}. \end{aligned}$$

Now,

$$\begin{aligned} \|\eta - \overline{R^{\mathrm{CFE}}}\eta\|_{L^\infty(0,T;L^2(\Omega))} &\leq \|\eta - \bar{\eta}\|_{L^\infty(0,T;L^2(\Omega))} + \|\bar{\eta} - \overline{R^{\mathrm{CFE}}}\eta\|_{L^\infty(0,T;L^2(\Omega))} \\ &\leq Ck^{1/2}\|\eta\|_{H^1(0,T;L^2(\Omega))} + CH\widetilde{\mathrm{Log}}^{1/2}(H/h)\|\bar{\eta}\|_{L^\infty(0,T;H^1(\Omega))} \\ &\leq C(k^{1/2} + H\widetilde{\mathrm{Log}}^{1/2}(H/h))\|\eta\|_{2,1} \\ &\leq C(k^{1/2} + H\widetilde{\mathrm{Log}}^{1/2}(H/h))\|f\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

Altogether these estimates yields

$$|Q_2| \leq C(H\widetilde{\mathrm{Log}}^{1/2}(H/h) + k^{1/2})\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}\|f\|_{L^2(0,T;L^2(\Omega))}. \tag{6.21}$$

Similarly, by the definition of P^{CFE} and (6.13), we have

$$\begin{aligned}
|Q_3| &= |(u_0 - U_0^{\mathrm{CFE}}, \eta(0))| \\
&= |(u_0 - P^{\mathrm{CFE}}u_0, \eta(0) - \eta^{\mathrm{CFE}}(0)) + (u_0 - P^{\mathrm{CFE}}u_0, \eta^{\mathrm{CFE}}(0))| \\
&= |(u_0 - P^{\mathrm{CFE}}u_0, \eta(0) - \eta^{\mathrm{CFE}}(0))| \\
&\leq C\|u_0\|_{0,\Omega}\|\eta(0) - \eta^{\mathrm{CFE}}(0)\| \\
&\leq CH\widetilde{\mathrm{Log}}^{1/2}(H/h)\|u_0\|_{0,\Omega}\|f\|_{L^2(0,T;L^2(\Omega))}. \qquad (6.22)
\end{aligned}$$

Combining (6.20), (6.21) and (6.22) with (6.18), we obtain

$$\begin{aligned}
\|u - U^{\mathrm{CFE}}\|_{L^2(0,T;L^2(\Omega))} &= \sup_{\substack{f\in L^2(0,T;L^2(\Omega))\\ f\neq 0}} \frac{(f, u - U^{\mathrm{CFE}})_{\Omega_T}}{\|f\|_{L^2(0,T;L^2(\Omega))}} \\
&\leq C(H\widetilde{\mathrm{Log}}^{1/2}(H/h) + k^{1/2})(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|u_0\|_{0,\Omega}),
\end{aligned}$$

and this completes the proof. □

Remark 6.3.2. *Theorem 6.3.2 shows an order $\mathcal{O}(H\widetilde{Log}^{1/2}(H/h) + k^{1/2})$ of convergence for the fully discrete case with respect to measure data in time. Note that, for the 'one-scale CFE method' i.e., when $h = \mathcal{O}(H)$, the result coincides with the standard FEM (cf. Gong [46]).*

6.4 CFE Error Estimates for Nonconvex Domains

In this section, we carry out the above semidiscrete and fully discrete error estimates to nonconvex domains. We show that there is a reduction in the convergence from the order $\mathcal{O}(H\widetilde{\mathrm{Log}}^{1/2}(H/h))$ in a convex polygonal domain, to $\mathcal{O}(H^s\widetilde{\mathrm{Log}}^{s/2}(H/h))$, $1/2 \leq s \leq 1$ in a nonconvex polygonal domain.

We now prove the following lemma.

Lemma 6.4.1. *Let $\xi, \eta \in \overline{X}(0,T) \hookrightarrow C([0,T]; H_0^1(\Omega))$ be the solutions of the problem (6.4) and (6.5), respectively. Also, let ξ^{CFE} and η^{CFE} be the solutions of (6.7) and (6.8), respectively, where $\xi^{CFE}, \eta^{CFE} \in H^1(0,T;S^{CFE})$. Then, for $\tilde{e} = \xi - \xi^{CFE}$ or $\tilde{e} = \eta - \eta^{CFE}$, we have*

$$\|\tilde{e}\|_{L^\infty(0,T;L^2(\Omega))} \leq CH^s\widetilde{Log}^{s/2}(H/h)\|f\|_{L^2(0,T;L^2(\Omega))}.$$

Proof. We recall equation (6.10) and use Lemma 6.2.3 to obtain

$$\begin{aligned}
&(\tilde{e}_t, \tilde{e}) + \mathcal{A}(\tilde{e}, \tilde{e}) \\
&\qquad \leq \|\tilde{e}_t\| \|R^{\mathrm{CFE}}\xi - \xi\| + C\|\tilde{e}\|_1 \|R^{\mathrm{CFE}}\xi - \xi\|_1 \\
&\qquad \leq C \left(H^s \widetilde{\mathrm{Log}}^{s/2}(H/h) \|\tilde{e}_t\| \right) \left(H^s \widetilde{\mathrm{Log}}^{s/2}(H/h) \|\xi\|_{1+s} \right) \\
&\qquad\qquad + C\|\tilde{e}\|_1 \left(H^s \widetilde{\mathrm{Log}}^{s/2}(H/h) \|\xi\|_{1+s} \right),
\end{aligned}$$

or,

$$\frac{d}{dt}\|\tilde{e}\|^2 + C_0\|\tilde{e}\|_1^2 \leq C H^{2s} \widetilde{\mathrm{Log}}^{s}(H/h)(\|\tilde{e}_t\|^2 + \|\xi\|_{1+s}^2),$$

after integration from 0 to t with $\tilde{e}(0) = 0$, we have

$$\|\tilde{e}(t)\|^2 + C_0 \int_0^t \|\tilde{e}\|_1^2 d\zeta \leq C H^{2s} \widetilde{\mathrm{Log}}^{s}(H/h) \int_0^t (\|\tilde{e}_\zeta\|^2 + \|\xi\|_{1+s}^2) d\zeta. \tag{6.23}$$

Now using the regularity estimates [8, 22], we have

$$\|\xi\|_{1+s} \leq \|\xi_t\| + \|f\|,$$

hence by standard estimates (cf. Lemma 6.2.1),

$$\int_0^t \|\xi\|_{1+s}^2 d\zeta \leq \int_0^t (\|\xi_\zeta\|^2 + \|f\|^2) d\zeta \leq C \int_0^T \|f\|^2 d\zeta.$$

Thus, (6.23) implies

$$\|\xi(t) - \xi^{\mathrm{CFE}}(t)\|^2 + C_0 \int_0^t \|\xi - \xi^{\mathrm{CFE}}\|_1^2 d\zeta \leq C H^{2s} \widetilde{\mathrm{Log}}^{s}(H/h) \int_0^T \|f\|^2 d\zeta.$$

Similarly, for $\tilde{e}(t) = \eta(t) - \eta^{\mathrm{CFE}}(t)$, we have

$$\|\eta(t) - \eta^{\mathrm{CFE}}(t)\|^2 + C_0 \int_0^t \|\eta - \eta^{\mathrm{CFE}}\|_1^2 d\zeta \leq C H^{2s} \widetilde{\mathrm{Log}}^{s}(H/h) \int_0^T \|f\|^2 d\zeta, \tag{6.24}$$

and this completes the proof. $\square$

Therefore, our semidiscrete error estimate can be stated as follows:

Theorem 6.4.1. *Assume $\mu = \sigma\omega$, σ and ω are given functions such that $\sigma \in C([0,T]; L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. Let $u \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ and $u^{CFE} \in L^2(0,T; S^{CFE})$ be the solutions of problems (6.2) and (6.3), respectively. Then, for $1/2 \leq s \leq 1$, we have the following error estimate:*

$$\|u - u^{CFE}\|_{L^2(0,T;L^2(\Omega))} \leq C H^s \widetilde{Log}^{s/2}(H/h)(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))} \|\omega\|_{\mathcal{M}[0,T]} + \|u_0\|_{0,\Omega}).$$

Proof. Let η be the solution of problem (6.5) with $f \in L^2(0,T;L^2(\Omega))$. Then a similar calculation as in Theorem 6.3.1 and an use of (6.24) yields

$$\begin{aligned}\int_{\Omega_T}(u-u^{\mathrm{CFE}})f\,dxdt &\leq C(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}\\ &\qquad + \|u_0\|_{0,\Omega})\|\eta-\eta^{\mathrm{CFE}}\|_{L^\infty(0,T;L^2(\Omega))}\\ &\leq CH^s\widetilde{\mathrm{Log}}^{s/2}(H/h)(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}+\|u_0\|_{0,\Omega})\|f\|_{L^2(0,T;L^2(\Omega))},\end{aligned}$$

which completes the proof of the theorem. $\square$

Now we turn to the fully discrete error estimate based on the $L^2(L^2)$-norm.

Theorem 6.4.2. *Assume $\mu=\sigma\omega$, σ and ω are given functions such that $\sigma \in C([0,T];L^2(\Omega))$ and $\omega \in \mathcal{M}[0,T]$. Let $u \in L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ and U^{CFE} be the solutions of the problems (6.2) and (6.15), respectively. Then, for $1/2 \leq s \leq 1$, we have*

$$\begin{aligned}&\|u-U^{CFE}\|_{L^2(0,T;L^2(\Omega))}\\ &\qquad\leq C(H^s\widetilde{Log}^{s/2}(H/h)+k^{1/2})(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}+\|u_0\|_{0,\Omega}).\end{aligned}$$

Proof. Following the similar argument as in Theorem 6.3.2, we obtain

$$\begin{aligned}&\int_{\Omega_T}(u-U^{\mathrm{CFE}})f\,dxdt\\ &\quad=-\sum_{i=1}^{N}\int_{J_i}\{k^{-1}(U_i^{\mathrm{CFE}}-U_{i-1}^{\mathrm{CFE}},\eta^{i-1}-\overline{R^{\mathrm{CFE}}}\eta)+\mathcal{A}(U_i^{\mathrm{CFE}},\bar{\eta}-\overline{R^{\mathrm{CFE}}}\eta)\}dt\\ &\quad+\{\langle\mu,\eta\rangle_{\Omega_T}-\sum_{i=1}^{N}\int_{J_i}\langle\mu,\overline{R^{\mathrm{CFE}}}\eta\rangle_{J_i}\}+(u_0-U_0^{\mathrm{CFE}},\eta(0)):=Q_1+Q_2+Q_3,\end{aligned}\tag{6.25}$$

where Q_i's are defined as in (6.18). In order to estimate Q_1, we have this time

$$\begin{aligned}&\|\eta^{i-1}-\overline{R^{\mathrm{CFE}}}\eta\|_{0,\Omega}\\ &\qquad\leq\|\eta^{i-1}-\bar{\eta}\|_{0,\Omega}+\|\bar{\eta}-\overline{R^{\mathrm{CFE}}}\eta\|_{0,\Omega}\\ &\qquad\leq k^{1/2}\|\eta_t\|_{L^2(t_{i-1},t_i;L^2(\Omega))}+CH^{2s}\widetilde{\mathrm{Log}}^{s}(H/h)\|\bar{\eta}\|_{1+s,\Omega}\\ &\qquad\leq k^{1/2}\|\eta_t\|_{L^2(t_{i-1},t_i;L^2(\Omega))}+CH^{2s}\widetilde{\mathrm{Log}}^{s}(H/h)k^{-1/2}\|\eta\|_{L^2(t_{i-1},t_i;H^{1+s}(\Omega))},\end{aligned}$$

and hence, using the fact $Ch^2 \leq CH^2 \leq CH^{2s}$ for $1/2 \leq s \leq 1$, the term G_2 is estimated as

$$\begin{aligned} G_2 &\leq C\left[\sum_{i=1}^{N}\left(k\|\eta_t\|^2_{L^2(t_{i-1},t_i;L^2(\Omega))} + H^{4s}\widetilde{\text{Log}}^{2s}(H/h)k^{-1}\|\eta\|^2_{L^2(t_{i-1},t_i;H^{1+s}(\Omega))}\right)\right]^{1/2} \\ &\leq C(H^s\widetilde{\text{Log}}^{s}(H/h) + k^{1/2})\|f\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

Therefore, (6.19) leads to

$$\begin{aligned} |Q_1| \leq C(H^s\widetilde{\text{Log}}^{s}(H/h) + k^{1/2})(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} \\ + \|u_0\|_{0,\Omega})\|f\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{6.26}$$

To estimate Q_2, we write

$$\begin{aligned} &\|\eta - \overline{R^{\text{CFE}}}\eta\|_{L^\infty(0,T;L^2(\Omega))} \\ &\quad \leq \|\eta - \bar{\eta}\|_{L^\infty(0,T;L^2(\Omega))} + \|\bar{\eta} - \overline{R^{\text{CFE}}}\eta\|_{L^\infty(0,T;L^2(\Omega))} \\ &\quad \leq Ck^{1/2}\|\eta\|_{H^1(0,T;L^2(\Omega))} + CH^s\widetilde{\text{Log}}^{s/2}(H/h)\|\bar{\eta}\|_{L^\infty(0,T;H^1(\Omega))} \\ &\quad \leq C(k^{1/2} + H^s\widetilde{\text{Log}}^{s/2}(H/h))\|\eta\|_{1+s,1} \\ &\quad \leq C(k^{1/2} + H^s\widetilde{\text{Log}}^{s/2}(H/h))\|f\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

Thus, we have

$$|Q_2| \leq C(k^{1/2} + H^s\widetilde{\text{Log}}^{s/2}(H/h))\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]}\|f\|_{L^2(0,T;L^2(\Omega))}. \tag{6.27}$$

Also, following the procedure as before for Q_3 and using (6.24), we have

$$\begin{aligned} |Q_3| &= |(u_0 - U_0^{\text{CFE}}, \eta(0))| \\ &\leq C\|u_0\|_{0,\Omega}\|\eta(0) - \eta^{\text{CFE}}(0)\| \\ &\leq CH^s\widetilde{\text{Log}}^{s/2}(H/h)\|u_0\|_{0,\Omega}\|f\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{6.28}$$

Altogether (6.25)-(6.28) leads to

$$\begin{aligned} &\|u - U^{\text{CFE}}\|_{L^2(0,T;L^2(\Omega))} \\ &\quad \leq C(H^s\widetilde{\text{Log}}^{s/2}(H/h) + k^{1/2})(\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\|\omega\|_{\mathcal{M}[0,T]} + \|u_0\|_{0,\Omega}), \end{aligned}$$

which completes the proof. □

Remark 6.4.1. *Theorems 6.4.1 and 6.4.2 yield error estimates of order* $\mathcal{O}(H^s\widetilde{Log}^{s/2}(H/h))$ *and* $\mathcal{O}(H^s\widetilde{Log}^{s/2}(H/h) + k^{1/2})$*,* $1/2 \leq s \leq 1$ *for the semidiscrete and the fully discrete scheme, respectively in a nonconvex domain. Due to the singularity in the re-entrant corners the order of convergence is reduced, however for* $s = 1$*, these results coincide with the previous results derived for the convex case (see Theorems 6.3.1 and 6.3.2).*

6.5 Concluding Remarks

In this chapter, we have undertaken CFEs for the discretization of parabolic equations in a two-dimensional convex as well as nonconvex domain with measure data in time. A continuous, piecewise linear CFE space is employed for the spatially semidiscrete finite element approximation and the temporal discretization is based on the backward Euler scheme. We derive *a priori* error estimates in the $L^2(L^2)$-norm for the spatially semidiscrete and fully discrete problems in both convex and nonconvex domains. Our analysis relies on the error estimates for the associate elliptic projection in the CFE framework and the duality argument. Our results not only generalize the work of [46] to CFE method for convex domains but also treat the problem (6.1) for nonconvex domains and derive convergence properties for CFE method. Parabolic equations with measure data in time finds huge applications in optimal control theory. For the CFE method, our estimated results in this chapter can be viewed as the first step, but a crucial step, for the error analysis of the CFE approximation to parabolic optimal control problems with pointwise control, where control acts on finitely many points of the domain. In the present approach, the degrees of freedom of the CFE space located on the inner region (Ω^{in}) of the domain, hence the minimal dimension of CFE space is independent of the domain geometry. Therefore, the dimension of the CFE space is less than the standard finite element space, and this is advantageous mostly for problems on domains with complicated geometry. An order $\mathcal{O}(H\widetilde{\text{Log}}^{1/2}(H/h) + k^{1/2})$ of convergence in the $L^2(L^2)$-norm is obtained for the fully discretization finite element approximation of parabolic equations with measure data in time for the convex polygonal domain, whereas the order reduces to $\mathcal{O}(H^s\widetilde{\text{Log}}^{s/2}(H/h) + k^{1/2})$, $1/2 \leq s \leq 1$ for the nonconvex polygonal domain with respect to the space discretization. Numerical results are presented in Chapter 7 for both convex and nonconvex domains (see Examples 7.6 and 7.7) to support our theoretical analysis.

7 Numerical Experiments

In this chapter, we shall present the results of numerical experiment to exemplify some of the practical aspects of *a priori* error estimates derived in the previous chapters. The main goal here is to compute the CFE solutions in every time level and evaluate the ROC. For the two-scale CFE discretization, we choose the coarse and the fine-scale mesh size and perform numerical tests on uniform time steps. The ROC will depend on the coarse-scale only. All computations are carried out using the software FreeFEM++ (cf. Hecht [55]). The computation of the errors for the backward Euler method of Chapter 2 in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms with smooth initial data are presented in Example 7.1. In Example 7.2, we compute the $L^\infty(L^2)$ errors and the corresponding ROC for nonsmooth initial data for the backward Euler method of Chapter 3. Examples 7.3 and 7.4 evaluate the CFE errors for both smooth and nonsmooth initial data for the *a priori* error estimates discussed in Chapter 4. The computation of the CFE solutions and the corresponding errors presented in Example 7.5 validates the theoretical results of Chapter 5. Finally, Examples 7.6 and 7.7 illustrate the theoretical results of Chapter 6 by computing the errors in the $L^2(L^2)$-norm for the solutions.

For a given finite number of successive iterations (indexed by i), let

- $E(i) :=$ the error corresponding to the $L^\infty(L^2)$, $L^\infty(H^1)$ or $L^2(L^2)$-norms on the i-th iteration, and
- $H(i) :=$ the corresponding coarse-scale mesh size.

Then the ROC is given by

$$\mathrm{ROC}(E(i)) = \frac{\log(E(i+1)/E(i))}{\log(H(i+1)/H(i))}. \tag{7.1}$$

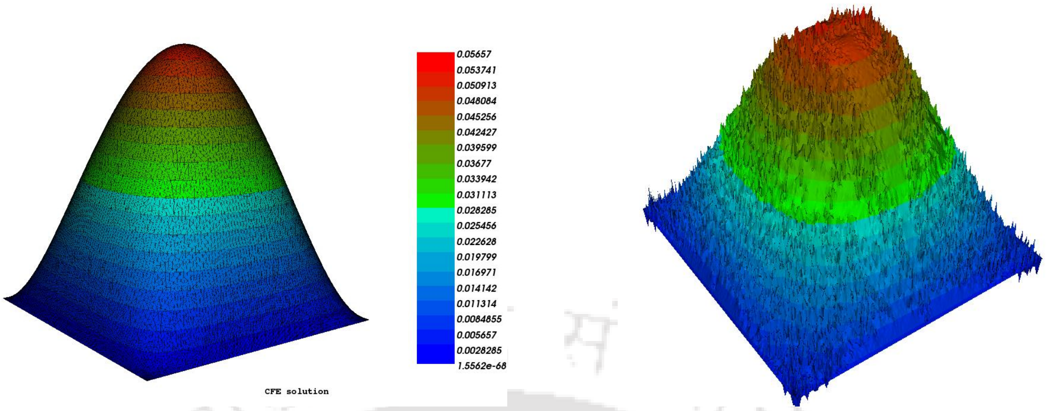
Example 7.1. This example is subjected to find the CFE solution and calculate the $L^\infty(L^2)$ and $L^\infty(H^1)$ errors for the coarse-scale grid of the backward Euler method presented in Theorem 2.4.2. Consider the following time dependent problem:

$$
\begin{aligned}
u_t - \Delta u &= -e^{-t}\{xy(x-1)(y-1) + 2x(x-1) + 2y(y-1)\} \quad \text{in } \Omega \times (0, 0.1], \\
u &= 0 \quad \text{on } \Gamma \times (0, 0.1], \qquad (7.2)\\
u(x, y, 0) &= xy(x-1)(y-1) \quad \text{in } \Omega,
\end{aligned}
$$

where $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. The exact solution of (7.2) is given by $u(x,y,t) = e^{-t}xy(x-1)(y-1)$.

We choose the coarse and fine-scale mesh size such that $h \ll H$ for the two-scale CFE method and the stopping criterion follows from (1.9). Initially, we choose the coarse mesh size $H = 0.152$, the finer mesh size $h = 0.072$ and the time step $k = 0.02$. We calculate the number of triangles in $\mathcal{T}_H^{\text{in}}$ as the inner grids containing the degrees of freedom. The CFE solution corresponding to smooth initial data is given by Figure 7.1a. Figure 7.1b shows the behaviour of the error as a function of (x, y). For each run, the spatial mesh size becomes half of the previous mesh size. Tables 7.1 – 7.4 compute the CFE solution and the corresponding errors in each time level. We denote the CFE error in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms by CFE_{err1} and CFE_{err2}, respectively. Table 7.5 reveals the second order convergence for the error in the $L^\infty(L^2)$-norm and linear order convergence in the $L^\infty(H^1)$-norm.

For the purpose of comparison of CFE method with the standard FEM we consider the same problem (7.2). For the standard FEM, we choose the initial mesh size $\hat{h} = 0.152$ and perform the numerical tests on the uniform time step $k = 0.02$. Let the errors in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms for the standard FEM be denoted by FE_{err1} and FE_{err2}, respectively. The computations for the standard FEM are shown in Table 7.6, which calculates the $\sharp$dof, the errors in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms and the corresponding ROC. From Tables 7.5 and 7.6 it is clear that the CFE method produces an optimal order convergence which coincides with the standard FEM, whereas the number of unknowns (i.e., $\sharp$dof) for the CFE method is much smaller than the standard FEM.



(a) (b)

Figure 7.1: *(a) CFE solution computed in domain Ω for smooth initial data at the time level $t = 0.1$ corresponding to the coarse-scale size $H = 0.019$ with the maximum value 0.05657. (b) CFE error for smooth initial data ($H = 0.019$).*

Table 7.1: *$H = 0.152$, $\sharp$ triangles in $\mathcal{T}_H^{in} = 156$ for Example 7.1*

Time level(t's)	u^{CFE}	$\lVert u^{\text{CFE}}(t) - u(t)\rVert_{L^2}$	$\lVert u^{\text{CFE}}(t) - u(t)\rVert_{H^1}$
0.02	0.0612773	6.06715e-04	1.64418e-02
0.04	0.0599843	6.05182e-04	1.61197e-02
0.06	0.0587474	6.01556e-04	1.58042e-02
0.08	0.0575514	5.89049e-04	1.54944e-02
0.10	0.0563894	5.66624e-04	1.51899e-02

Table 7.2: *$H = 0.076$, $\sharp$ triangles in $\mathcal{T}_H^{in} = 590$ for Example 7.1*

Time level(t's)	u^{CFE}	$\lVert u^{\text{CFE}}(t) - u(t)\rVert_{L^2}$	$\lVert u^{\text{CFE}}(t) - u(t)\rVert_{H^1}$
0.02	0.0612158	1.48133e-04	8.65068e-03
0.04	0.0599861	1.47746e-04	8.34819e-03
0.06	0.0587868	1.46899e-04	8.08527e-03
0.08	0.057615	1.43885e-04	7.92549e-03
0.10	0.0564687	1.38452e-04	7.7688e-03

Table 7.3: $H = 0.038$, $\sharp$ *triangles in* $\mathcal{T}_H^{in} = 2426$ *for Example 7.1*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$	$\|u^{\text{CFE}}(t) - u(t)\|_{H^1}$
0.02	0.0612495	3.3209e-05	4.51563e-03
0.04	0.0600351	3.3090e-05	4.23216e-03
0.06	0.0588453	3.2819e-05	4.05034e-03
0.08	0.0576795	3.2441e-05	3.97015e-03
0.10	0.0565369	3.1988e-05	3.89154e-03

Table 7.4: $H = 0.019$, $\sharp$ *triangles in* $\mathcal{T}_H^{in} = 9760$ *for Example 7.1*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$	$\|u^{\text{CFE}}(t) - u(t)\|_{H^1}$
0.02	0.0612623	6.95739e-06	2.28019e-03
0.04	0.0600514	5.95558e-06	2.18803e-03
0.06	0.0588638	5.29591e-06	2.0467e-03
0.08	0.0576994	4.85273e-06	2.00619e-03
0.10	0.0565676	4.54462e-06	1.96648e-03

Table 7.5: *CFE errors in the* $L^\infty(L^2)$ *and* $L^\infty(H^1)$*-norms and the ROC for Example 7.1*

H	$\sharp$ dof	CFE_{err1}	ROC	CFE_{err2}	ROC
0.152	95	6.06715e-04	—	1.64418e-02	—
0.076	328	1.48133e-04	2.03	8.65068e-03	0.926
0.038	1278	3.3209e-05	2.16	4.51563e-03	0.938
0.019	5009	6.95739e-06	2.25	2.28019e-03	0.986

Table 7.6: *Finite element errors in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms and the ROC for Example 7.1*

$\hat{h}$	♯ dof	FE_{err1}	ROC	FE_{err2}	ROC
0.152	244	5.41466e-04	—	1.43544e-02	—
0.076	954	1.00817e-04	2.43	5.38731e-03	1.41
0.038	3818	2.03482e-05	2.31	2.61137e-03	1.04
0.019	14062	4.57315e-06	2.15	1.30817e-03	0.997

Example 7.2. In this example, we study the numerical behaviour of the errors in the $L^\infty(L^2)$-norm and compute the corresponding ROC for the backward Euler method presented in Theorem 3.2.2. Consider the following test problem:

$$
\begin{aligned}
u_t - \frac{1}{12}\Delta u &= 0 \quad \text{in } \Omega \times (0, 0.5],\\
u &= 0 \quad \text{on } \Gamma \times (0, 0.5],\\
u(x, y, 0) &= u_0(x, y) \quad \text{in } \Omega,
\end{aligned} \tag{7.3}
$$

where $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and

$$
u_0(x, y) = \begin{cases} 1 & \text{if } 1/4 \le x, y \le 3/4,\\ 0 & \text{otherwise.} \end{cases}
$$

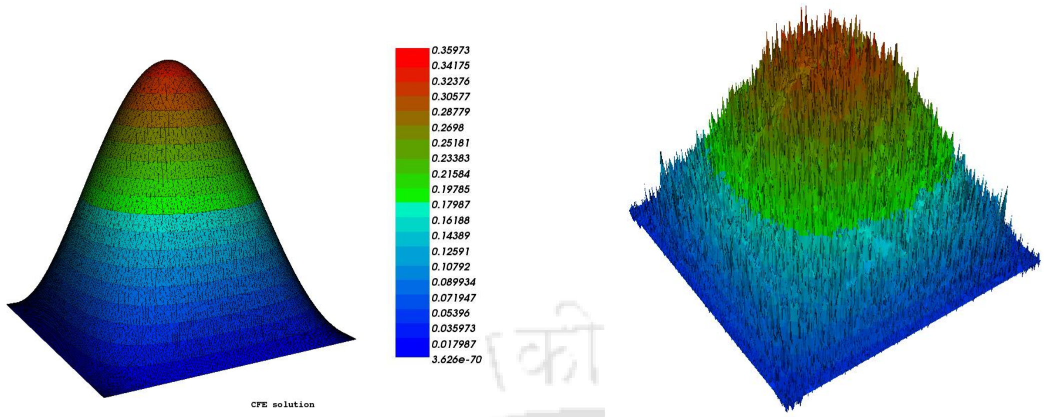
The exact solution of (7.3) is given by

$$
u(x, y, t) = \frac{8}{\pi^2} \sum_{i,j=0}^{\infty} c_i c_j \exp\Big(-\pi^2 t \frac{(2i+1)^2 + (2j+1)^2}{12} \Big) \times \sin(\pi x(2i+1)) \sin(\pi y(2j+1)),
$$

where

$$
c_i = \begin{cases} (-1)^{(i/2)}(2i+1)^{-1} & \text{if } i \text{ is even},\\ (-1)^{(i+1)/2}(2i+1)^{-1} & \text{otherwise.} \end{cases}
$$

Initially, we choose the coarse mesh size $H = 0.32$, the finer mesh size $h = 0.148$ and the time step size $k = 0.1$. The CFE solution corresponding to given nonsmooth initial data



(a) (b)

Figure 7.2: *(a) CFE solution computed in domain Ω for nonsmooth initial data at the time level $t = 0.5$ corresponding to the coarse-scale size $H = 0.04$ with the maximum value 0.35973. (b) CFE error for nonsmooth initial data ($H = 0.04$).*

shown in Figure 7.2a and the behaviour of the error as a function of (x, y) is depicted in Figure 7.2b. We use the same formula given by (7.1) to compute the ROC. Tables 7.7 – 7.10 compute the CFE solutions and the corresponding errors in various time level. The computation shows an optimal order convergence for $t > 0$ even for nonsmooth initial data (see Table 7.11).

We also consider the standard FEM for the problem (7.3). Choosing the initial mesh size $\hat{h} = 0.32$ we perform the numerical tests on the uniform time steps $k = 0.1$. The results for numerical simulation are given in Table 7.12. The $\sharp$ dof, errors in the $L^\infty(L^2)$-norm and the corresponding ROC are calculated in Table 7.12. Tables 7.11 and 7.12 show that the $\sharp$ dof for the standard FEM is more than the CFE method.

Table 7.7: *$H = 0.32$, $\sharp$ triangles in $\mathcal{T}_H^{in} = 36$ for Example 7.2*

Time level(t's)	u^{CFE}	$\lVert u^{\text{CFE}}(t) - u(t) \rVert_{L^2}$
0.1	0.802554	7.61202e-01
0.2	0.530683	2.32296e-01
0.3	0.389728	8.7469e-02
0.4	0.305853	7.3275e-02
0.5	0.248969	6.2003e-02

Table 7.8: $H = 0.16$, $\sharp$ *triangles in* $\mathcal{T}_H^{in} = 152$ *for Example 7.2*

Time level(t's)	u^{CFE}	$\|u^{\mathrm{CFE}}(t) - u(t)\|_{L^2}$
0.1	0.851426	1.98382e-01
0.2	0.625978	9.7096e-02
0.3	0.479243	4.4589e-02
0.4	0.383026	3.4541e-02
0.5	0.314538	2.8232e-02

Table 7.9: $H = 0.08$, $\sharp$ *triangles in* $\mathcal{T}_H^{in} = 584$ *for Example 7.2*

Time level(t's)	u^{CFE}	$\|u^{\mathrm{CFE}}(t) - u(t)\|_{L^2}$
0.1	0.872603	5.4658e-02
0.2	0.666299	4.2113e-02
0.3	0.519585	2.6926e-02
0.4	0.419023	1.7194e-02
0.5	0.345704	1.2538e-02

Table 7.10: $H = 0.04$, $\sharp$ *triangles in* $\mathcal{T}_H^{in} = 2398$ *for Example 7.2*

Time level(t's)	u^{CFE}	$\|u^{\mathrm{CFE}}(t) - u(t)\|_{L^2}$
0.1	0.882367	1.45424e-02
0.2	0.683835	1.13604e-02
0.3	0.537399	1.04583e-02
0.4	0.435109	8.0191e-03
0.5	0.359734	6.2444e-03

Table 7.11: *CFE errors in the $L^\infty(L^2)$-norm and the ROC for Example 7.2*

H	$\sharp$ dof	CFE_{err1}	ROC
0.32	27	7.61202e-01	—
0.16	93	1.98382e-01	1.94
0.08	325	5.4658e-02	1.86
0.04	1264	1.45424e-02	1.91

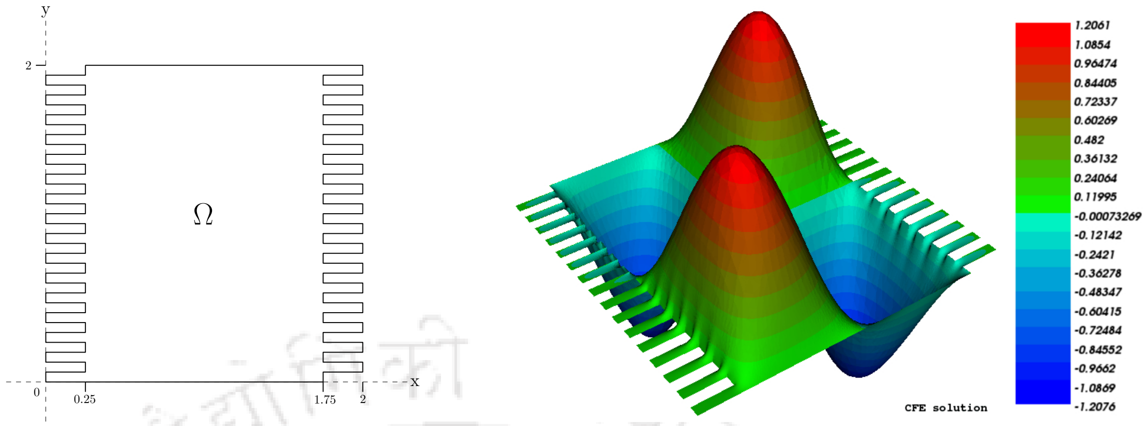
Table 7.12: *Finite element errors in the $L^\infty(L^2)$-norm and the ROC for Example 7.2*

$\hat{h}$	$\sharp$ dof	FE_{err1}	ROC
0.32	56	4.18136e-01	—
0.16	240	1.02771e-01	2.03
0.08	945	2.59921e-02	1.98
0.04	3798	6.76534e-03	1.94

Example 7.3. In this example, our aim is to solve the given problem (7.4) numerically to find the CFE solution for the backward Euler method presented in Theorem 4.2.2 and calculate the $L^\infty(L^2)$ and $L^\infty(H^1)$ errors for the coarse-scale grid and the corresponding ROC which depends on the coarse-scale only. We consider a computational domain with a complicated boundary in this example. Let the domain $\Omega = (0,2) \times (0,2) \subset \mathbb{R}^2$, where a series of tiny "finger-like" cuts have been removed from both the left and right-hand boundary. More precisely, each side of the domain $x = 0$, $0 \le y \le 2$ and $x = 2$, $0 \le y \le 2$, possesses 16 equidistributed tiny gaps (cf. Figure 7.3a). Consider the following problem in Ω:

$$
\begin{aligned}
u_t - \Delta u &= (2\pi^2 + \cos(t))\, e^{\sin(t)} \sin(\pi x) \sin(\pi y) && \text{in } \Omega \times (0, 0.5], \\
u &= 0 && \text{on } \Gamma \times (0, 0.5], \qquad (7.4) \\
u(x,y,0) &= \sin(\pi x) \sin(\pi y) && \text{in } \Omega.
\end{aligned}
$$

We discretized the domain Ω by employing two-scale grid, where H be the coarse-scale and h be the finer scale mesh size. Initially, we choose the coarse mesh size



(a) (b)

Figure 7.3: *(a) Square-shaped domain with a series of tiny "finger-like" equidistributed cuts have been removed from both the left and right-hand boundary. (b) CFE solution computed in domain* Ω *for smooth initial data at the time level* $t = 0.5$ *corresponding to the coarse-scale size* $H = 0.04$ *with the maximum value* 1.20611.

$H = 0.32$, the finer mesh size $h = 0.1178$ and the time step $k = 0.1$. In order to compute the $\sharp$dof, we calculate the number of triangles in $\mathcal{T}_H^{\text{in}}$. The experiment is carried out with $\mathbb{P}_1$ elements. The CFE solution corresponding to the smooth initial data is depicted in Figure 7.3b. The ROC is calculated by the formula (7.1). The CFE solutions and the errors for different coarse-scale mesh size H in various time levels are computed in Tables 7.13 – 7.16. We investigate the $\sharp$dof, errors and the corresponding ROC in both $L^\infty(L^2)$ and $L^\infty(H^1)$-norms. Table 7.17 reveals that numerical results validate our analytical ROC.

Table 7.13: *Coarse-scale size* $H = 0.32$ *for Example 7.3*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$	$\|u^{\text{CFE}}(t) - u(t)\|_{H^1}$
0.1	0.85022	1.62507e−01	8.79537e−01
0.2	0.88415	1.57349e−01	8.64399e−01
0.3	0.92778	1.52343e−01	8.47990e−01
0.4	0.99834	1.49461e−01	8.37386e−01
0.5	1.09645	1.34501e−01	8.26771e−01

Table 7.14: *Coarse-scale size* $H = 0.16$ *for Example 7.3*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$	$\|u^{\text{CFE}}(t) - u(t)\|_{H^1}$
0.1	0.86875	4.38051e−02	4.67244e−01
0.2	0.89045	4.28408e−02	4.45869e−01
0.3	0.96243	4.24893e−02	4.20463e−01
0.4	1.05173	4.13885e−02	4.13501e−01
0.5	1.14851	3.99385e−02	4.02826e−01

Table 7.15: *Coarse-scale size* $H = 0.08$ *for Example 7.3*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$	$\|u^{\text{CFE}}(t) - u(t)\|_{H^1}$
0.1	0.89847	1.14686e−02	2.46682e−01
0.2	0.93022	1.02814e−02	2.45910e−01
0.3	1.00744	9.84355e−03	2.42684e−01
0.4	1.10145	9.71614e−03	2.39461e−01
0.5	1.20305	9.55241e−03	2.37538e−01

Table 7.16: *Coarse-scale size* $H = 0.04$ *for Example 7.3*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$	$\|u^{\text{CFE}}(t) - u(t)\|_{H^1}$
0.1	0.89980	2.92015e−03	1.27353e−01
0.2	0.93221	2.87651e−03	1.11405e−01
0.3	1.00991	2.83209e−03	9.88414e−02
0.4	1.10423	2.80378e−03	9.44558e−02
0.5	1.20611	2.77881e−03	9.18084e−02

Table 7.17: *CFE errors in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms and the ROC for Example 7.3*

H	$\sharp$ dof	CFE_{err1}	ROC	CFE_{err2}	ROC
0.32	84	1.62507e−01	—	8.79537e−01	—
0.16	357	4.38051e−02	1.891	4.67244e−01	0.913
0.08	1386	1.14686e−02	1.933	2.46682e−01	0.922
0.04	5327	2.92015e−03	1.974	1.27353e−01	0.954

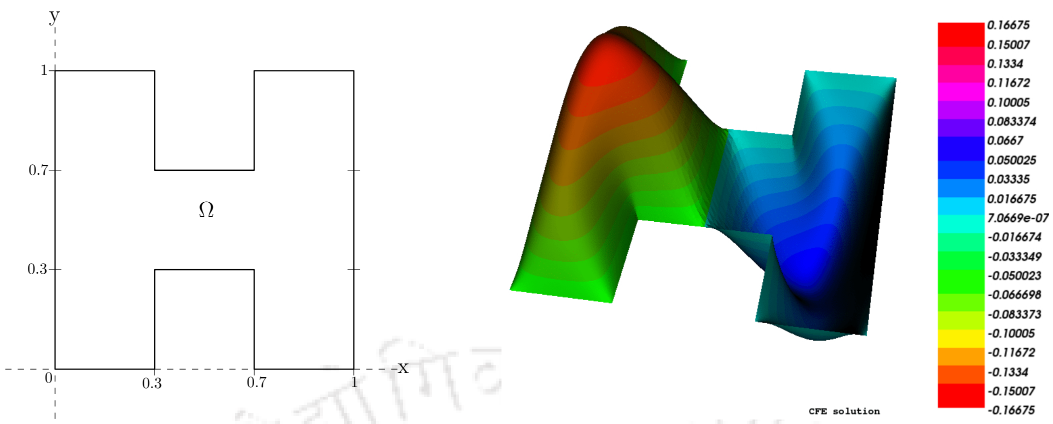
Example 7.4. This example is considered to evaluate the $L^\infty(L^2)$ error for the given problem (7.5) with nonsmooth initial data for the fully discrete backward Euler method presented in Theorem 4.3.2. We consider Ω as the H-shaped domain shown in Figure 7.4a. Consider the following test problem in Ω:

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } \Omega \times (0, 0.5], \\ u &= 0 && \text{on } \Gamma \times (0, 0.5], \\ u(x, y, 0) &= u_0(x, y) && \text{in } \Omega, \end{aligned} \tag{7.5}$$

where

$$u_0(x, y) = \begin{cases} 1 & \text{if } x \le 0.5 \quad \forall y \in \Omega, \\ -1 & \text{if } x > 0.5 \quad \forall y \in \Omega. \end{cases}$$

To validate the sharpness of the theoretical results we perform the numerical tests by choosing two-scale, coarse-scale (H) and fine-scale (h). Initially, we choose the coarse-scale size $H = 0.3$ and the fine-scale size $h = 0.12$. We perform the numerical tests on the uniform time step $k = 0.1$. The spatial mesh size becomes half of the previous mesh size in each run. The CFE solution is computed with the $\mathbb{P}_1$ finite elements and is evaluated in different time levels are shown in Tables 7.18 – 7.21. Figure 7.4b demonstrates the CFE solution corresponding to the nonsmooth initial data. Table 7.22 computes the $\sharp$ dof, errors and the corresponding ROC in the $L^\infty(L^2)$-norm for the H-shaped domain with nonsmooth initial data.



(a) (b)

Figure 7.4: *(a) H-shaped domain. (b) CFE solution computed in domain Ω for nonsmooth initial data at the time level $t = 0.5$ corresponding to the coarse-scale size $H = 0.0375$ with the maximum value* 0.16675.

Table 7.18: *Coarse-scale size $H = 0.3$ for Example 7.4*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$
0.1	0.177673	6.03407e−02
0.2	0.175656	6.02635e−02
0.3	0.172516	5.95997e−02
0.4	0.170365	5.38475e−02
0.5	0.169355	4.88781e−02

Table 7.19: *Coarse-scale size $H = 0.15$ for Example 7.4*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$
0.1	0.177409	1.83594e−02
0.2	0.175391	1.26481e−02
0.3	0.172253	9.86612e−03
0.4	0.169112	9.37742e−03
0.5	0.168539	8.78467e−03

Table 7.20: *Coarse-scale size $H = 0.075$ for Example 7.4*

Time level(t's)	u^{CFE}	$\|u^{\mathrm{CFE}}(t) - u(t)\|_{L^2}$
0.1	0.176524	5.13645e−03
0.2	0.174507	4.48228e−03
0.3	0.171372	3.81033e−03
0.4	0.168852	3.15085e−03
0.5	0.167954	2.84619e−03

Table 7.21: *Coarse-scale size $H = 0.0375$ for Example 7.4*

Time level(t's)	u^{CFE}	$\|u^{\mathrm{CFE}}(t) - u(t)\|_{L^2}$
0.1	0.174261	1.42837e−03
0.2	0.173244	1.00339e−03
0.3	0.171109	9.73796e−04
0.4	0.168223	9.32573e−04
0.5	0.166748	8.95397e−04

Table 7.22: *CFE errors in the $L^\infty(L^2)$-norm and the ROC for Example 7.4*

H	$\sharp$ dof	$\mathrm{CFE}_{\mathrm{err1}}$	ROC
0.3000	44	6.03407e−02	—
0.1500	155	1.83594e−02	1.717
0.0750	496	5.13645e−03	1.838
0.0375	1766	1.42837e−03	1.846

Example 7.5. Our purpose is to compute the CFE solution for the problem (7.6) and calculate the corresponding errors in the $L^\infty(L^2)$-norm to evaluate the ROC presented in Theorem 5.2.3. Let $\Omega = (0,1)^2 \setminus [0.6,1)^2$. Consider the following IBVP:

$$
\begin{aligned}
u_t - \nabla\cdot(a(u)\nabla u) &= 1 \quad \text{in } \Omega\times(0,0.1],\\
u &= 0 \quad \text{on } \Gamma\times(0,0.1],\\
\text{with } u(x,y,0) &= xy \quad \text{in } \Omega,
\end{aligned}
\tag{7.6}
$$

where, $a(u) = k_0(1+\beta u)$, $k_0 = 0.2$, $\beta = 0.002$. To compute the solution numerically we choose the mesh size corresponding to the coarser scale as $H = 0.152$ and the finer scale as $h = 0.072$. The temporal discretization is based on the linearized backward Euler method. We choose the uniform time step $k = 0.02$. Tables 7.23 – 7.26 computes the CFE solution in different time level for the $\mathbb{P}_1$ finite elements. It is clear from Table 7.27 that the CFE method yields an almost optimal order convergence for the given test problem (7.6) in an L-shaped domain.

Table 7.23: *Coarse-scale size* $H = 0.152$ *for Example 7.5*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$
0.02	0.410369	0.175232
0.04	0.305549	0.155759
0.06	0.264802	0.150272
0.08	0.246841	0.145008
0.10	0.245309	0.139832

Table 7.24: *Coarse-scale size* $H = 0.076$ *for Example 7.5*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$
0.02	0.382372	0.050974
0.04	0.321401	0.048779
0.06	0.297965	0.046450
0.08	0.272995	0.042963
0.10	0.261469	0.039374

Table 7.25: *Coarse-scale size* $H = 0.038$ *for Example 7.5*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$
0.02	0.374979	0.014511
0.04	0.325051	0.012593
0.06	0.293313	0.009869
0.08	0.276252	0.008679
0.10	0.260983	0.008305

Table 7.26: *Coarse-scale size* $H = 0.019$ *for Example 7.5*

Time level(t's)	u^{CFE}	$\|u^{\text{CFE}}(t) - u(t)\|_{L^2}$
0.02	0.373033	0.0042705
0.04	0.324247	0.0036182
0.06	0.294701	0.0032389
0.08	0.274014	0.0020663
0.10	0.259475	0.0013944

Table 7.27: *CFE errors in the $L^\infty(L^2)$-norm and the ROC for Example 7.5*

H	$\sharp$ dof	$\mathrm{CFE}_{\mathrm{err1}}$	ROC
0.152	72	0.175232	—
0.076	284	0.050974	1.78
0.038	1081	0.014511	1.81
0.019	2692	0.0042705	1.76

In the next two examples, we compute the CFE errors for parabolic problems with measure data in time for both convex and nonconvex domains, respectively. We consider the following parabolic equation with Dirac source term on the right-hand side in time:

$$\begin{aligned} u_t - \Delta u &= \mu \quad \text{in } \Omega_T, \\ u &= 0 \quad \text{on } \Gamma_T, \\ u(\cdot,0) &= u_0 \quad \text{in } \Omega, \end{aligned}$$

where $\mu = \sigma(x,t)\omega(t)$, $\sigma(x,t) \in C([0,T]; L^2(\Omega))$ and $\omega(t) \in \mathcal{M}[0,T]$. For a fixed time steps we first examine the behaviour of the error for a sequence of discretizations with various mesh sizes. Then we show the behaviour of the error for different time steps with a fixed spatial triangulation.

Example 7.6. In this example, we study the behaviour of the fully discrete backward Euler errors in the $L^2(L^2)$-norm presented in Theorem 6.3.2. Let $\Omega_T = \Omega \times [0,1]$, where $\Omega = (-1,1) \times (-1,1) \subset \mathbb{R}^2$. Consider the exact solution such that

$$\mu(x,t) = \sin(\pi|x|^2) \cdot \delta_t(\frac{1}{2}) + \sin(\pi|x|^2) \cdot \gamma(t) + (-4\pi \cos(\pi|x|^2) + 4\pi^2|x|^2 \sin(\pi|x|^2)) \cdot \beta(t),$$

where

$$\gamma(t) = \begin{cases} 2t, & t < 0.5, \\ 2t+2, & t \geq 0.5, \end{cases} \quad \text{and} \quad \beta(t) = \begin{cases} t^2, & t < 0.5, \\ t^2+2t, & t \geq 0.5, \end{cases}$$

and $\delta_t(z)$ denotes the Dirac measure with respect to the variable t concentrated at $t = z$ and the initial condition $u_0 = 0$.

We discretized the domain Ω by employing the two-scale grid, where H is chosen to be the coarse-scale mesh size and h is the fine-scale mesh size. Since the exact solution u is not known, thus the error is computed in standard way by taking the differences between the numerical solutions U^{CFE} for different mesh size. We test the convergence order with respect to space and time discretization in order to confirm the theoretical ROC. In order to investigate the convergence order with respect to space discretization we fixed the time discretization with $N = 2187$. Furthermore, to investigate the convergence order with respect to the time discretization we fixed the space discretization with 6945 degrees of freedom for the CFE space. We denote the CFE errors and the standard finite element errors in the $L^2(L^2)$-norm by CFE_{err3} and FE_{err3}, respectively.

Observing Table 7.28 we conclude that the ROC with respect to the space discretization is $\mathcal{O}(H^2\widetilde{\text{Log}}(H/h))$ and with respect to the time discretization is $\mathcal{O}(k)$. Both the results are better than our theoretical ROC. Table 7.29 computes the ROC for the standard FEM by choosing the same mesh size ($\hat{h} = H$, where $\hat{h}$ be the mesh size for the standard FEM), and note that the $\sharp$dof in the standard FEM is higher than the CFE method.

Table 7.28: *CFE errors in the $L^2(L^2)$-norm and the ROC for Example 7.6*

$\sharp$dof	CFE_{err3}	ROC	N	CFE_{err3}	ROC
9	8.452876e-01	—	3	1.654967e-01	—
43	2.597449e-01	1.702	9	8.903991e-02	0.894
130	8.471876e-02	1.616	27	4.759459e-02	0.904
401	2.969053e-02	1.513	81	2.512674e-02	0.921
1820	1.039381e-02	1.514	243	1.335980e-02	0.911
6945	3.657804e-03	1.507	729	7.318611e-03	0.868

Table 7.29: *Finite element errors in the $L^2(L^2)$-norm and the ROC for Example 7.6*

$\sharp$ dof	FE_{err3}	ROC	N	FE_{err3}	ROC
22	3.295312e-01	—	3	2.534983e-02	—
70	7.669150e-02	2.103	9	1.282108e-02	0.983
263	3.801804e-02	1.012	27	6.448768e-03	0.991
998	9.609069e-03	1.984	81	3.189907e-03	1.016
3831	2.515786e-03	1.933	243	1.608414e-03	0.988
15129	6.408696e-04	1.973	729	8.085935e-04	0.992

Example 7.7. In this example, we perform the numerical experiments to compute the CFE errors for the backward Euler method presented in Theorem 6.4.2. Let $\Omega_T = \Omega \times [0,1]$, where Ω is a H-shaped domain as depicted in Figure 7.4a. Assume $\gamma \in (0,1)$, $\lambda \in \mathbb{R}$ and $\epsilon(t) = e^{-\lambda t} - e^{-\lambda/2}$. Consider the exact solution such that

$$\mu(x,t) = \sin(\pi|x|^2)\cdot\delta(t) + \Bigg[\sin(\pi|x|^2)\cdot\frac{e^{\lambda t}}{1-\gamma} + \big\{-4\pi\cos(\pi|x|^2) + 4\pi^2|x|^2\sin(\pi|x|^2)\big\}\cdot\frac{e^{\lambda t}}{\lambda(1-\gamma)}\Bigg].\vartheta(t),$$

where

$$\delta(t) = \begin{cases} 0, & t \geq \frac{1}{2}, \\ \epsilon(t)^{-\gamma}, & t < \frac{1}{2}, \end{cases} \quad \text{and} \quad \vartheta(t) = \begin{cases} \epsilon(0)^{1-\gamma}, & t \geq \frac{1}{2}, \\ \epsilon(0)^{1-\gamma} - \epsilon(t)^{1-\gamma}, & t < \frac{1}{2}, \end{cases}$$

and the initial condition is chosen as $u_0 = 0$.

Here we investigate the theoretical ROC with respect to space and time discretization, respectively. For $\lambda = 1$, 2, we fixed the time discretization with $N = 2048$ in order to compute the convergence order with respect to the space discretization, while the space discretization is fixed with 5741 degrees of freedom to investigate the convergence order with respect to the time discretization. Tables 7.30 and 7.31 show the results for $\lambda = 1$ and $\lambda = 2$, respectively. From the results listed in Tables 7.30 and 7.31 it is clear that, the convergence order with respect to the space discretization is 1 and with respect to the time discretization is $\mathcal{O}(k^{1/2})$, which is consistent with our theoretical results.

Table 7.30: *CFE errors in the $L^2(L^2)$-norm and the ROC for Example 7.7 with $\lambda = 1$*

$\sharp$ dof	CFE_{err3}	ROC	N	CFE_{err3}	ROC
7	2.057311951	—	2	3.201419e-01	—
27	9.588952e-01	1.101	4	2.060825e-01	0.635
107	4.807041e-01	0.996	8	1.341176e-01	0.620
378	2.416223e-01	0.992	16	8.958405e-02	0.582
1636	1.214987e-01	0.992	32	5.964293e-02	0.587
5741	6.136635e-02	0.985	64	3.997911e-02	0.577

Table 7.31: *CFE errors in the $L^2(L^2)$-norm and the ROC for Example 7.7 with $\lambda = 2$*

$\sharp$ dof	CFE_{err3}	ROC	N	CFE_{err3}	ROC
7	2.941392317	—	2	4.139383e-01	—
27	1.345564269	1.128	4	2.790946e-01	0.569
107	6.550968e-01	1.038	8	1.951306e-01	0.516
378	3.286618e-01	0.995	16	1.395739e-01	0.483
1636	1.664162e-01	0.982	32	9.944764e-02	0.489
5741	8.457827e-02	0.976	64	7.141553e-02	0.478

Numerical experiments are presented in this chapter for the various kind of parabolic problems discussed in Chapters 2 – 6 in order to validate the theoretical results. An optimal order convergence in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms are presented for smooth initial data (cf. Table 7.5). Also for nonsmooth initial data, optimal order convergence is shown in the $L^\infty(L^2)$-norm (cf. Table 7.11). A comparison between the CFE method and the standard FEM (see, Tables 7.5 and 7.6 for smooth initial data, and Tables 7.11 and 7.12 for nonsmooth initial data) in terms of $\sharp$ dof is presented, which confirms that our proposed method provides a good approximate solution without increasing so much degrees of freedom. The computation of the errors and ROC for linear parabolic problems in complicated domains are presented for both smooth and nonsmooth initial data (cf. Tables 7.17 and 7.22). An almost optimal order convergence for nonlinear parabolic problems in nonconvex domains in the $L^\infty(L^2)$-norm has been established (cf. Table 7.27). Finally, numerical experiments are presented in order to compute the CFE errors and the corresponding ROC for parabolic problems with measure data in time for both convex (cf. Table 7.28) and nonconvex domains (cf. Tables 7.30 and 7.31). Further, a comparison between the CFE method and the standard FEM in terms of $\sharp$ dof has been presented for the convex domains (cf. Tables 7.28 and 7.29) which indicates the $\sharp$ dof of the CFE space is much smaller than the standard finite element space.

8 Conclusions and Extensions

The present chapter deals with the critical assessment of the results highlighting the significance of the current thesis work and the corresponding techniques used in deriving them. It also provides information for the scope of possible extensions and future investigations.

8.1 Critical Review of the Results

In this thesis, the study was set out to explore the *a priori* error analysis of the CFE methods for various kinds of parabolic problems as described in Chapters 2 – 6 in both convex and nonconvex domains. *A priori* error bounds for both smooth and nonsmooth initial data cases are considered and analyzed. We believe that the work in this thesis could be a first step, but a crucial step for the error analysis of the CFE method for various kind of IBVPs with complex geometry. The proposed method leads to an efficient numerical method that uses two-scale grids for the space discretization. The coarse-scale is used to approximate the solution in the interior of the domain at a proper distance to the boundary and the fine-scale is used to approximate the Dirichlet boundary conditions. In the CFE method, the degrees of freedom located in the inner region (Ω^{in}) of the domain, and hence, the minimal dimension of CFE space does not dependent on the domain geometry. Without much increment of degrees of freedom, the CFE method provides us a suitable technique to resolve the Dirichlet boundary conditions, which is very advantageous in contrast to the standard FEM, especially for problems on domains with complicated geometry. In order to estimates the error by an error splitting technique, we have introduced the elliptic or Ritz projection in this thesis. The estimates for the Ritz projection are derived in the CFE framework which plays an important role in the error analysis for both convex and nonconvex domains. The critical review of the results of each chapter is presented below.

Chapter 2 studies *a priori* error analysis for the spatially semidiscrete and fully discrete approximation for parabolic problem (2.1) in a bounded convex polygonal domain in $\mathbb{R}^2$ with smooth initial data. The essential tool used in the error analysis is the estimates for the Ritz projection in the CFE framework (cf. Lemma 2.4.1). For the spatially semidiscrete error analysis the splitting technique have been used, where an intermediate object is used to split the total error into two parts, namely the parabolic error and the elliptic error. Then, the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms *a priori* error bounds for the main error (cf. Theorem 2.4.1) are obtained by estimating each term separately. *A priori* error bounds for the fully discrete backward Euler approximation with constant and variable time steps (cf. Theorems 2.4.2 and 2.4.3, respectively) and for the Crank-Nicolson approximation (cf. Theorem 2.4.4) are derived .

In Chapter 3, we have studied *a priori* error analysis for the spatially semidiscrete and fully discrete approximation of (3.1) in a convex polygonal domain with nonsmooth initial data, i.e., $u_0 \in L^2(\Omega)$. The essential components used in the analysis are: (i) The eigenfunctions expansion related to the elliptic operator in the spatially semidiscrete error analysis, and (ii) the rational approximations for exponential in the fully discrete error analysis. A continuous, piecewise linear CFE space is employed for the spatially semidiscrete finite element approximation and the temporal discretization is based on the backward Euler scheme. Nonsmooth data error estimates in the $L^\infty(L^2)$-norm for positive time is established for both the spatially semidiscrete (cf. Theorem 3.2.1) and fully discrete backward Euler scheme (cf. Theorem 3.2.2).

A priori error analysis for the spatially semidiscrete and fully discrete schemes for the linear parabolic IBVP (4.1) in a nonconvex polygonal domain in $\mathbb{R}^2$ with smooth and nonsmooth initial data are discussed in Chapter 4. We have derived *a priori* error bounds for the Ritz projection (cf. Lemma 4.2.1) in the CFE framework when the exact solution u is having a lower regularity, i.e., $u \in H_0^1(\Omega) \cap H^{1+s}(\Omega)$, $1/2 \le s \le 1$. The estimates for the spatially semidiscrete error in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms with smooth initial data (cf. Theorem 4.2.1) and in the $L^\infty(L^2)$-norm with nonsmooth initial data (cf. Theorem 4.3.1) are established. Moreover, *a priori* error analysis for the fully discrete backward Euler time discretization scheme for both constant and variable time steps with smooth initial data (cf. Theorems 4.2.2 and 4.2.3) and with nonsmooth initial data (cf. Theorem 4.3.2) are presented.

Chapter 5 deals with the *a priori* error analysis for the nonlinear parabolic IBVPs (5.1) concerning the spatially semidiscrete (cf. Theorem 5.2.1) and fully discrete (cf. Theorems 5.2.2 and 5.2.3) schemes. An almost optimal order convergence (up to logarithmic terms) in space and optimal order in time for the CFE method in the $L^\infty(L^2)$-

norm are demonstrated. The spatially semidiscrete and fully discrete error estimates rely on the bounds of $\tilde{\rho}$ and $\tilde{\rho}_t$ in the $L^\infty(L^2)$ and $L^\infty(H^1)$-norms (cf. Lemmas 5.2.2 and 5.2.3). The backward Euler method is used in the fully discrete error analysis. We have also considered the modified linearized backward Euler method for the time discretization in order to avoid the laborious task of solving a nonlinear system of algebraic equations in every time level.

Chapter 6 is devoted to study both the spatially semidiscrete and fully discrete CFE approximations of parabolic problem (6.1) with measure data in time for both convex and nonconvex domains. The space discretization uses the continuous, piecewise linear CFEs while the backward Euler approximation is used for the time discretization. The key argument used in the error estimates is the estimates for the associated elliptic projection in the CFE framework in both convex and nonconvex domains and the duality treatment. The main crucial part in the error analysis is to estimate the errors for the dual problem with respect to the $L^\infty(L^2)$-norm (cf. Lemmas 6.3.1 and 6.4.1). For both convex and nonconvex domains, the error estimates for the spatially semidiscrete method (cf. Theorems 6.3.1 and 6.4.1) and the fully discrete backward Euler method (cf. Theorems 6.3.2 and 6.4.2) in the $L^2(L^2)$-norm are derived.

Chapter 7 is concerned with numerical assessment of the derived error estimates of Chapters 2 – 6. All computations are carried out using the software FreeFEM++ [55]. For the two-scale grid discretization in space, the coarse-scale (H) and the fine-scale (h) is chosen. The main emphasis is given on the computation of the CFE solutions in different time level, the errors in various norms and the corresponding ROC. The comparison between the CFE method and the standard FEM is shown in terms of $\sharp$ dof, which reveals that the CFE method enables us to provide a good approximate solution with much less $\sharp$ dof than the standard FEM.

8.2 Extensions and Remarks

The CFE method for parabolic problems in convex and nonconvex domains have a wide variety of applications in science and engineering. Despite being so rich in the numerical approximations to solutions of parabolic IBVPs and the importance of the CFE method, the topic of CFE approximations for parabolic equations in a convex and nonconvex polygonal domain are yet to be explored. To the best of our knowledge, this is the first instance that *a priori* error estimates of CFE method for the space-time discretization of various kind of parabolic problems are attempted. Moreover, the proposed method developed in this thesis unlock several new research directions. Now, we briefly outline some interesting problems that may be undertaken and studied in near future.

Two-scale CFE method for an evolution equation with a positive-type memory term in both convex and nonconvex domains with smooth initial data. The CFE error analysis developed in this thesis can be extended to an evolution equation with a positive-type memory term. We consider the model IBVPs of the form

$$
\begin{aligned}
u_t(t) - \int_0^t \bar{\beta}(t-\zeta)\Delta u(\zeta)d\zeta &= f(t) \quad \text{in } \Omega,\ t \geq 0,\\
u &= 0 \quad \text{on } \Gamma,\ t > 0,\\
u(\cdot,0) &= u_0 \quad \text{in } \Omega,
\end{aligned} \tag{8.1}
$$

where $u_t = \partial u/\partial t$, the initial data u_0 is a function of x only is assumed to be smooth. We consider Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary Γ, where the domain Ω may be convex as well as nonconvex. Here $\bar{\beta}$ is a real-valued and positive-definite kernel, i.e., for each $t \in (0,T]$ with $T > 0$, $\bar{\beta} \in L_{1,\text{loc}}(\overline{\mathbb{R}}_+)$, $\mathbb{R}_+ = (0,\infty)$, and satisfies

$$\int_0^T \varphi(t) \int_0^t \bar{\beta}(t-\zeta)\varphi(\zeta)d\zeta dt \geq 0 \quad \forall \varphi \in C[0,T].$$

McLean and Thomée [71] have studied the numerical solution of an IBVP for a Volterra type integro-differential equation of the form (8.1), in which the integral operator is a convolution product of a positive-definite kernel and an elliptic partial differential operator. They have used Galerkin FEM in order to discretize the equation in space and used finite differences in combination with various quadrature rules (which preserve the positive character of the memory term) to discretize the equation in time. An obvious extension of our work is to study *a priori* error analysis for the spatially semidiscrete and fully discrete approximations of (8.1). We believe that the estimates for the elliptic or Ritz projection in the CFE framework for both convex and nonconvex domains will play a vital role in the error analysis.

Two-scale CFE method for an evolution equation with a positive-type memory term in both convex and nonconvex domains with nonsmooth initial data. Consider the IBVP (8.1) with homogeneous form ($f = 0$) as

$$
\begin{aligned}
u_t(t) - \int_0^t \bar{\beta}(t-\zeta)\Delta u(\zeta)d\zeta &= 0 \quad \text{in } \Omega,\ t \geq 0,\\
u &= 0 \quad \text{on } \Gamma,\ t > 0,\\
u(\cdot,0) &= u_0 \quad \text{in } \Omega,
\end{aligned} \tag{8.2}
$$

where $\Omega \subset \mathbb{R}^2$ may be a convex (or nonconvex) polygonal domain and $\bar{\beta}$ is a real-valued and positive-definite kernel. The given initial data assumed to be nonsmooth, i.e., $u_0 \in L^2(\Omega)$.

Some investigations concerning nonsmooth data error analysis for an approximations of an evolution equation with a positive-type memory term have been conducted in the framework of FEM by Lubich *et al.* [67]. It will be an interesting and challenging task to extend nonsmooth data error analysis for the CFE method for both convex and nonconvex domains as discussed in Chapters 3 and 4 to the equation (8.2) with a positive-type memory term.

Two-scale CFE method for parabolic interface problems in both convex and nonconvex domains. Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$ with Lipschitz boundary Γ, and let Ω_1 be a subdomain of Ω with C^2 boundary $\Gamma_1 := \widetilde{\Gamma}$. The interface $\widetilde{\Gamma}$ divides the domain Ω into two subdomains Ω_1 and $\Omega_2 := \Omega \backslash \Omega_1$. We consider the linear parabolic interface problem of the form

$$u_t(x,t) - \operatorname{div}(\tilde{\beta}(x)\nabla u(x,t)) = f(x,t) \quad \text{in } \Omega \times (0,T] \tag{8.3}$$

with prescribed initial and boundary conditions

$$u = 0 \quad \text{on } \Gamma \times [0,T]; \qquad u(x,0) = u_0(x) \quad \text{in } \Omega \tag{8.4}$$

and the jump conditions on the interface

$$[u] = 0, \qquad \left[\tilde{\beta}\frac{\partial u}{\partial \mathbf{n}}\right] = 0 \quad \text{across } \widetilde{\Gamma} \times [0,T], \tag{8.5}$$

where $u_t = \frac{\partial u}{\partial t}$, $[v]$ denotes the jump of a quantity v across the interface $\widetilde{\Gamma}$, i.e., $[v](x) = v_1(x) - v_2(x)$, $x \in \widetilde{\Gamma}$ with $v_i(x) = v(x)|_{\Omega_i}$ for $i = 1,2$, and $T < +\infty$, and $\mathbf{n}$ denotes the unit outward normal to the boundary $\Gamma_1 := \widetilde{\Gamma}$. The initial data $u_0(x)$ and the forcing term $f(x,t)$ are real-valued functions and are assumed to be smooth. Moreover, the interface is assumed to be of arbitrary shape with class of C^2, and the diffusion coefficient $\tilde{\beta}(x)$ is assumed to be positive and piecewise constant on each subdomain, i.e.,

$$\tilde{\beta}(x) = \tilde{\beta}_i \quad \text{for } x \in \Omega_i, \quad i = 1,2.$$

Recently, Peterseim [76] has developed the CFE method for elliptic interface problems. The author has considered CFEs to approximate the Dirichlet problem with discontinuous diffusion coefficient, where the discontinuity appears across the interface of the domains. To the best of authors' knowledge, the CFE method for parabolic interface problems in convex domains are not available in the literature. It would be interesting to investigate the CFE error analysis for parabolic interface problem (8.3) – (8.5) in convex domains. It would be further challenging to study the convergence analysis of the CFE method for parabolic interface problem (8.3) – (8.5) in a nonconvex polygonal domain Ω with the interface $\widetilde{\Gamma}$ is of class C^2.

A posteriori error analysis of the CFE method for parabolic problems in both convex and nonconvex domains. An *a posteriori* error estimate is a computable quantity in terms of the approximate solution and the data of the given problem, i.e., *a posteriori* error estimators employ the numerical solution and the data of the problem itself to derive estimates on the actual errors. A relatively complete theory for the derivation of *a posteriori* error estimators of FEM for elliptic and parabolic problems has been developed by Babuška and Rheinboldt [7], Verfürth [103], Eriksson and Johnson [36] and Picasso [78]. On the contrary to *a priori* error analysis, *a posteriori* error analysis predicts the bounds of the form

$$\|u - U\|_{\mathcal{X}} \leq \widehat{\eta}(U, data),$$

where $\|\cdot\|_{\mathcal{X}}$ denotes a specified norm. The estimator $\widehat{\eta}(U, data)$ is a computable quantity which depends on the numerical solution U and the data of the problem, and also $\widehat{\eta}(U, data)$ decreases with respect to the mesh parameters requiring the lowest possible regularity permitted by the problem. *A posteriori* error analysis of CFE method for elliptic PDEs on domains with complicated structures has been extensively studied by Carstensen and Sauter [17]. As a model problem, they have considered the Poisson equation with mixed boundary conditions on a domain which contains huge number of geometric details such as small holes or pores, and presented a reliable and efficient *a posteriori* error estimates. Thus, another obvious extension of our work would be to extend *a posteriori* error analysis for the CFE method for elliptic problems to parabolic problems in both convex and nonconvex domains.

CFE approximations with higher degree polynomials. The extension of CFE error analysis with higher degree polynomials for both elliptic and parabolic problems is an interesting issue and will be taken up in future.

Bibliography

[1] V. M. Abdullayev and K. R. Aida-zade, *Finite-difference methods for solving loaded parabolic equations*, Comput. Math. Math. Phys., 56 (2016), pp. 93–105.

[2] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[3] R. A. Adams and J. J. Fournier, *Sobolev Spaces*, vol. 140, Academic Press, Amsterdam, 2003.

[4] P. F. Antonietti, S. Giani, and P. Houston, *hp-version composite discontinuous Galerkin methods for elliptic problems on complicated domains*, SIAM J. Sci. Comput., 35 (2013), pp. A1417–A1439.

[5] G. P. Astrakhantsev, *A finite difference method for solving the third boundary value problem for elliptic and parabolic equations in an arbitrary region. Iterative solution of finite difference equations. I*, USSR Comput. Math. Math. Phys., 11 (1971), pp. 141–161.

[6] I. Babuška, *Finite element method for domains with corners*, Computing, 6 (1970), pp. 264–273.

[7] I. Babuška and W. C. Rheinboldt, *A posteriori error estimates for the finite element method*, Int. J. Numer. Meth. Engng., 12 (1978), pp. 1597–1615.

[8] C. Bacuta, J. Bramble, and J. Xu, *Regularity estimates for elliptic boundary value problems in Besov spaces*, Math. Comp., 72 (2003), pp. 1577–1595.

[9] C. Bacuta, J. H. Bramble, and J. E. Pasciak, *New interpolation results and applications to finite element methods for elliptic boundary value problems*, East-West J. Numer. Math., 9 (2001), pp. 179–198.

[10] R. E. Bank and J. Xu, *A Hierarchical Basis Multigrid Method for Unstructured Grids, In: W. Hackbusch, G. Wittum, eds.*, Fast Solvers for Flow Problems, Proceedings of the Tenth GAMM-Seminar, Kiel. Wiesbaden: Vieweg, 1995.

[11] ——, *An algorithm for coarsening unstructured meshes*, Numer. Math., 73 (1996), pp. 1–36.

[12] J. Blair, *Approximate solution of elliptic and parabolic boundary value problems*, PhD thesis, Univ. of California, Berkeley, (1970).

[13] L. Boccardo, A. Dall'Aglio, T. Gallouët, and L. Orsina, *Nonlinear parabolic equations with measure data*, J. Funct. Anal., 147 (1997), pp. 237–258.

[14] L. Boccardo and T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal., 87 (1989), pp. 149–169.

[15] J. H. Bramble, J. E. Pasciak, P. H. Sammon, and V. Thomée, *Incomplete iterations in multistep backward difference methods for parabolic problems with smooth and nonsmooth data*, Math. Comp., 52 (1989), pp. 339–367.

[16] S. C. Brenner and R. Scott, *The Mathematical Theory of Finite Element Methods*, vol. 15, Texts Appl. Math., Springer, New York, third ed., 2008.

[17] C. Carstensen and S. A. Sauter, *A posteriori error analysis for elliptic PDEs on domains with complicated structures*, Numer. Math., 96 (2004), pp. 691–721.

[18] E. Casas, *L^2 estimates for the finite element method for the Dirichlet problem with singular data*, Numer. Math., 47 (1985), pp. 627–632.

[19] ——, *Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations*, SIAM J. Control Optim., 35 (1997), pp. 1297–1327.

[20] T. F. Chan and B. F. Smith, *Domain decomposition and multigrid algorithms for elliptic problems on unstructured meshes*, Contemp. Math., 2 (1994), pp. 171–182.

[21] P. Chatzipantelidis, R. Lazarov, and V. Thomée, *Parabolic finite volume element equations in nonconvex polygonal domains*, Numer. Methods Partial Differential Eq., 25 (2009), pp. 507–525.

[22] P. Chatzipantelidis, R. Lazarov, V. Thomée, and L. Wahlbin, *Parabolic finite element equations in nonconvex polygonal domains*, BIT Numer. Math., 46 (2006), pp. S113–S143.

[23] P. Chatzipantelidis and R. D. Lazarov, *Error estimates for a finite volume element method for elliptic PDEs in nonconvex polygonal domains*, SIAM J. Numer. Anal., 42 (2005), pp. 1932–1958.

[24] K. Chrysafinos and L. S. Hou, *Error estimates for semidiscrete finite element approximations of linear and semilinear parabolic equations under minimal regularity assumptions*, SIAM J. Numer. Anal., 40 (2002), pp. 282–306.

[25] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, vol. 40, Class. Appl. Math., SIAM, Philadelphia, 2002.

[26] B. D. Coleman, *On thermodynamics, strain impulses, and viscoelasticity*, Arch. Rational Mech. Anal., 17 (1964), pp. 230–254.

[27] B. D. Coleman and M. E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys., 18 (1967), pp. 199–208.

[28] M. Crouzeix and V. Thomée, *Resolvent estimates in l_p for discrete Laplacians on irregular meshes and maximum-norm stability of parabolic finite difference schemes*, Comput. Methods Appl. Math., 1 (2001), pp. 99–117.

[29] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Evolution Problems, I*, vol. 5, Springer-Verlag, Berlin, 1992.

[30] K. Deckelnick and M. Hinze, *Variational discretization of parabolic control problems in the presence of pointwise state constraints*, J. Comput. Math., 29 (2011), pp. 1–15.

[31] J. Dendy, Jr., *Galerkin's method for some highly nonlinear problems*, SIAM J. Numer. Anal., 14 (1977), pp. 327–347.

[32] J. Douglas, Jr., *Effective time-stepping methods for the numerical solution of nonlinear parabolic problems*, The Math. Finite Elem. Appl. III, (1979), pp. 289–304.

[33] J. Douglas, Jr. and T. Dupont, *Galerkin methods for parabolic equations*, SIAM J. Numer. Anal., 7 (1970), pp. 575–626.

[34] J. Douglas, Jr. and T. Dupont, *Galerkin methods for parabolic equations with nonlinear boundary conditions*, Numer. Math., 20 (1973), pp. 213–237.

[35] ——, *The effect of interpolating the coefficients in nonlinear parabolic Galerkin procedures*, Math. Comp., 29 (1975), pp. 360–389.

[36] K. Eriksson and C. Johnson, *Adaptive finite element methods for parabolic problems. I. A linear model problem*, SIAM J. Numer. Anal., 28 (1991), pp. 43–77.

[37] K. Eriksson, C. Johnson, and S. Larsson, *Adaptive finite element methods for parabolic problems VI: Analytic semigroups*, SIAM J. Numer. Anal., 35 (1998), pp. 1315–1325.

[38] L. C. Evans, *Partial Differential Equations*, vol. 19, Graduate Studies in Mathematics, Amer. Math. Society, RI, 1998.

[39] I. Faragó, S. Korotov, and P. Neittaanmäki, *Galerkin approximations for the linear parabolic equation with the third boundary condition*, Appl. Math., 48 (2003), pp. 111–128.

[40] G. Fix and N. Nassif, *On finite element approximations to time-dependent problems*, Numer. Math., 19 (1972), pp. 127–135.

[41] H. Fujita and A. Mizutani, *On the finite element method for parabolic equations, I; approximation of holomorphic semigroups*, J. Math. Soc. Japan, 28 (1976), pp. 749–771.

[42] T. Gallouët and R. Herbin, *Convergence of linear finite elements for diffusion equations with measure data*, C. R. Math. Acad. Sci. Paris Sér, 338 (2004), pp. 81–84.

[43] H. Gao, B. Li, and W. Sun, *Stability and convergence of fully discrete Galerkin FEMs for the nonlinear thermistor equations in a nonconvex polygon*, Numer. Math., 136 (2017), pp. 383–409.

[44] T. Gao, W. H. Zhang, J. H. Zhu, Y. J. Xu, and D. H. Bassir, *Topology optimization of heat conduction problem involving design-dependent heat load effect*, Finite Elem. Anal. Des., 44 (2008), pp. 805–813.

[45] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, vol. 5, Springer Series Comput. Math., Springer-Verlag, Berlin, 1986.

[46] W. Gong, *Error estimates for finite element approximations of parabolic equations with measure data*, Math. Comp., 82 (2013), pp. 69–98.

[47] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, vol. 69, Class. Appl. Math., Pitman, Boston, 1985.

[48] ——, *Singularities in Boundary Value Problems*, Masson, Paris, 1992.

[49] G. J. Habetler and R. L. Schiffman, *A finite differnce method for analyzing the compression of poro-viscoelastic media*, Computing, 6 (1970), pp. 342–348.

[50] W. Hackbusch and S. A. Sauter, *Adaptive composite finite elements for the solution of PDEs containing nonuniformely distributed micro-scales*, Mater. Model., 8 (1996), pp. 31–43.

[51] ——, *Composite finite elements for problems containing small geometric details. Part II: Implementation and numerical results*, Comput. Vis. Sci., 1 (1997), pp. 15–25.

[52] ——, *Composite finite elements for the approximation of PDEs on domains with complicated micro-structures*, Numer. Math., 75 (1997), pp. 447–472.

[53] W. Hackbusch and S. A. Sauter, *A new finite element approach for problems containing small geometric details*, Arch. Math., 34 (1998), pp. 105–117.

[54] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988.

[55] F. Hecht, *New development in FreeFem++*, J. Numer. Math., 20 (2012), pp. 251–266.

[56] R. B. Kellogg, *Interpolation between subspaces of a Hilbert space, Tech. Note BN-719*, University of Maryland, College Park, (1971).

[57] R. B. Kellogg, T.-Y. Li, and J. Yorke, *A constructive proof of the Brouwer fixed-point theorem and computational results*, SIAM J. Numer. Anal., 13 (1976), pp. 473–483.

[58] R. Kornhuber and H. Yserentant, *Multilevel methods for elliptic problems on domains not resolved by the coarse grid*, Contemp. Math., 180 (1994), pp. 49–49.

[59] V. Kosmopoulos and T. S. Keller, *Damage-based finite-element vertebroplasty simulations*, Eur. Spine J., 13 (2004), pp. 617–625.

[60] V. A. Kozlov, V. G. Mazia, and J. Rossmann, *Elliptic Boundary Value Problems in Domains with Point Singularities*, vol. 52, Amer. Math. Society, Providence, RI, 1997.

[61] J. R. Kweon, *The evolution compressible Navier-Stokes system on polygonal domains*, J. Differ. Equ., 232 (2007), pp. 487–520.

[62] ——, *Regularity of solutions for the Navier-Stokes system of incompressible flows on a polygon*, J. Differ. Equ., 235 (2007), pp. 166–198.

[63] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, vol. 23, Amer. Math. Society, Providence, 1968.

[64] B. Li and Z. Zhang, *Mathematical and numerical analysis of the time-dependent Ginzburg–Landau equations in nonconvex polygons based on Hodge decomposition*, Math. Comp., 86 (2017), pp. 1579–1608.

[65] J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, 1971.

[66] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Springer-Verlag, Berlin, 1972.

[67] C. H. Lubich, I. H. Sloan, and V. Thomée, *Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term*, Math. Comp., 65 (1996), pp. 1–17.

[68] M. Luskin, *A Galerkin method for nonlinear parabolic equations with nonlinear boundary conditions*, SIAM J. Numer. Anal., 16 (1979), pp. 284–299.

[69] M. Luskin and R. Rannacher, *On the smoothing property of the Crank-Nicolson scheme*, Appl. Anal., 14 (1982), pp. 117–135.

[70] M. Luskin and R. Rannacher, *On the smoothing property of the Galerkin method for parabolic equations*, SIAM J. Numer. Anal., 19 (1982), pp. 93–113.

[71] W. McLean and V. Thomée, *Numerical solution of an evolution equation with a positive-type memory term*, J. Austral. Math. Soc. Ser., 35 (1993), pp. 23–70.

[72] D. Meidner, R. Rannacher, and B. Vexler, *A priori error estimates for finite element discretizations of parabolic optimization problems with pointwise state constraints in time*, SIAM J. Control Optim., 49 (2011), pp. 1961–1997.

[73] A. M. Oberman and I. Zwiers, *Adaptive finite difference methods for nonlinear elliptic and parabolic partial differential equations with free boundaries*, J. Sci. Comput., 68 (2016), pp. 231–251.

[74] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44, Appl. Math. Sci., Springer, New York, 1983.

[75] A. Pegoretti, L. Fambri, G. Zappini, and M. Bianchetti, *Finite element analysis of a glass fibre reinforced composite endodontic post*, Biomaterials, 23 (2002), pp. 2667–2682.

[76] D. Peterseim, *Composite finite elements for elliptic interface problems*, Math. Comp., 83 (2014), pp. 2657–2674.

[77] D. Peterseim and S. A. Sauter, *The composite mini element–coarse mesh computation of Stokes flows on complicated domains*, SIAM J. Numer. Anal., 46 (2008), pp. 3181–3206.

[78] M. Picasso, *Adaptive finite elements for a linear parabolic problem*, Comput. Methods Appl. Mech. Engrg., 167 (1998), pp. 223–237.

[79] T. Preusser, M. Rumpf, S. A. Sauter, and L. O. Schwen, *3D composite finite elements for elliptic boundary value problems with discontinuous coefficients*, SIAM J. Sci. Comput., 33 (2011), pp. 2115–2143.

[80] H. S. Price and R. S. Varga, *Error bounds for semidiscrete Galerkin approximations of parabolic problems with applications to petroleum reservoir mechanics*, Numerical Solution of Field Problems in Continuum Physics, 2 (1970), pp. 74–94.

[81] F. U. Prieto, J. J. B. Muñoz, and L. G. Corvinos, *Application of the generalized finite difference method to solve the advection–diffusion equation*, J. Comput. Appl. Math., 235 (2011), pp. 1849–1855.

[82] H. Rachford, Jr., *Two-level discrete-time Galerkin approximations for second order nonlinear parabolic partial differential equations*, SIAM J. Numer. Anal., 10 (1973), pp. 1010–1026.

[83] M. Rama Mohana Rao, *Ordinary Differential Equations: Theory and Applications*, East-West Press Pvt. Ltd., New Delhi, 1980.

[84] G. Raugel, *Résolution numérique par une méthode d'élements finis du problème de Dirichlet pour le laplacien dans un polygone*, C. R. Acad. Sci. Paris, Sér. A, 286 (1977), pp. 791–794.

[85] M. Rech, *Composite finite elements: An adaptive two-scale approach to the nonconforming approximation of Dirichlet problems on complicated domains*, PhD thesis, Universität Zürich, (2006).

[86] M. Rech, S. A. Sauter, and A. Smolianski, *Two-scale composite finite element method for Dirichlet problems on complicated domains*, Numer. Math., 102 (2006), pp. 681–708.

[87] J. Ruge and K. Stüben, *Algebraic Multigrid, in Multigrid Methods*, SIAM, Philadelphia, 1987.

[88] P. Sammon, *Convergence estimates for semidiscrete parabolic equation approximations*, SIAM J. Numer. Anal., 19 (1983), pp. 68–92.

[89] L. Sapa, *A finite difference method for quasi-linear and nonlinear differential functional parabolic equations with Neumann's condition*, Comment. Math., 1 (2009), pp. 83–106.

[90] S. A. Sauter, *On extension theorems for domains having small geometric details*, tech. rep., 96-03, University of Kiel, Kiel, Germany, 1996.

[91] S. A. Sauter and R. Warnke, *Extension operators and approximation on domains containing small geometric details*, East-West J. Numer. Math., 7 (1999), pp. 61–77.

[92] S. A. Sauter and R. Warnke, *Composite finite elements for elliptic boundary value problems with discontinuous coefficients*, Computing, 77 (2006), pp. 29–55.

[93] A. H. Schatz, V. Thomée, and L. B. Wahlbin, *Maximum norm stability and error estimates in parabolic finite element equations*, Commun. Pure Appl. Math., 33 (1980), pp. 265–304.

[94] A. H. Schatz and L. B. Wahlbin, *Maximum norm estimates in the finite element method on plane polygonal domains, Part 2, Refinements*, Math. Comp., 32 (1978), pp. 73–109.

[95] L. O. Schwen, T. Preusser, and M. Rumpf, *Composite finite elements for 3D elasticity with discontinuous coefficients*, in Proceedings of the 16th Workshop on the Finite Element Method in Biomedical Engineering, Biomechanics and Related Fields. University of Ulm, Germany, 2009.

[96] B. Sinsoysal and T. Coruhlu, *Finite differences method for one dimensional nonlinear parabolic equation with double nonlinearity*, in Int. Conf. Numer. Anal. Appl., Springer, 2012, pp. 495–501.

[97] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

[98] H. B. Stewart, *Generation of analytic semigroups by strongly elliptic operators*, Trans. Amer. Math. Soc., 199 (1974), pp. 141–162.

[99] V. Thomée, *Some convergence results for Galerkin methods for parabolic boundary value problem*, in Mathematical Aspects of Finite Elements, C. de Boor, ed., Academic Press, New York, (1974), pp. 55–88.

[100] V. Thomée, *Negative norm estimates and superconvergence in Galerkin methods for parabolic problems*, Math. Comp., 34 (1980), pp. 93–113.

[101] V. Thomée, *Finite difference methods for linear parabolic equations*, in P. G. Ciarlet and J.-L. Lions, eds., Handbook of Numer. Anal., *I*, North-Holland, Amsterdam, 1 (1990), pp. 5–196.

[102] ——, *Galerkin Finite Element Methods for Parabolic Problems*, vol. 1054, Springer, Berlin, 2006.

[103] R. Verfürth, *A Review of A Posteriori Error Estimation and Adaptive Mesh Refinement Techniques*, Wiley-Teubner, New York, 1996.

[104] M. F. Wheeler, *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal., 10 (1973), pp. 723–759.

[105] M. Zlàmal, *Finite element methods for nonlinear parabolic equations*, RAIRO Modél. Math. Anal. Numér., 11 (1977), pp. 93–107.

List of published and communicated papers

Based on the work carried out in this thesis, the following published and communicated papers have resulted:

1. T. Pramanick and R. K. Sinha, *Two-scale composite finite element method for parabolic problems with smooth and nonsmooth initial data*, J. Appl. Math. Comput., 58 (2017), pp. 469–501, DOI: 10.1007/s12190-017-1153-9.

2. T. Pramanick and R. K. Sinha, *Composite finite element approximation for nonlinear parabolic problems in nonconvex polygonal domains*, Numer. Methods for Partial Differential Eq., 34 (2018), pp. 1–20, DOI: 10.1002/num.22293.

3. T. Pramanick and R. K. Sinha, *Composite finite element approximation for parabolic problems in nonconvex polygonal domains*, Comput. Methods Appl. Math., (2019), DOI: 10.1515/cmam-2018-0155.

4. T. Pramanick and R. K. Sinha, *Error estimates for two-scale composite finite element approximations of parabolic equations with measure data in time for convex and nonconvex polygonal domains*, Submitted for Publication.